Algebraic surfaces

Lecture I: The Picard group, Riemann-Roch,...

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Divisors and line bundles

Surface = smooth, projective, over $\mathbb{C}$.

$\mathrm{Pic}(S) = \{\text{line bundles on } S\}/\sim$, (group for $\otimes$).

$\mathrm{Div}(S) = \{D = \sum n_i C_i\}$. $D \geqslant 0$ (effective) if $n_i \geqslant 0\ \forall i$.

$$\{D \geqslant 0\} \ \overset{\sim}{\leftrightsquigarrow}\ \{(L,s) \,|\, L \in \mathrm{Pic}(S), 0 \neq s \in H^0(L)\}$$

We put $L = \mathcal{O}_S(D)$. Map $D \mapsto \mathcal{O}_S(D)$ extends by linearity to homomorphism $\mathrm{Div}(S) \twoheadrightarrow \mathrm{Pic}(S)$. Then $\mathrm{Pic}(S) = \mathrm{Div}(S)/\equiv$ where $D \equiv D' \Leftrightarrow D - D' = \mathrm{div}(\varphi)$, φ rational function on S.

C irreducible curve, $s \in H^0(\mathcal{O}_S(C))$ defining C. $\mathcal{O}_S(-C) \xhookrightarrow{s} \mathcal{O}_S$
$\Rightarrow$ $\mathcal{O}_S(-C) \cong$ ideal sheaf of C in S.

$f : S \to T \ \rightsquigarrow\ f^* : \mathrm{Pic}(T) \to \mathrm{Pic}(S)$.
$D \in \mathrm{Div}(T)$; if $f(S) \not\subset D$, $f^*D \in \mathrm{Div}(S)$ and $\mathcal{O}_S(f^*D) = f^*\mathcal{O}_S(D)$.

The intersection form

$C \neq D$ irreducible, $p \in C \cap D$. f, g equations of C, D in $\mathcal{O}_p$.

Definition : $m_p(C \cap D) := \dim_{\mathbb{C}} \mathcal{O}_p/(f,g)$.

Example: $m_p(C \cap D) = 1 \iff (f,g) = \mathfrak{m}_p \iff f, g$ local coordinates at $p \stackrel{\text{def}}{\iff} C$ and D transverse.

Definition : $(C \cdot D) := \displaystyle\sum_{p \in C \cap D} m_p(C \cap D)$.

Theorem

$\exists$ *bilinear symmetric form* $(\ \cdot\ ) : \mathrm{Pic}(S) \times \mathrm{Pic}(S) \to \mathbb{Z}$ *such that* $(\mathcal{O}_S(C) \cdot \mathcal{O}_S(D)) = (C \cdot D)$ *for* C, D *irreducible.*

Remark : Suppose C smooth, $D \geqslant 0$. $\mathcal{O}_S(D)$ has a section s with $\mathrm{div}(s) = D$; then $(C \cdot D) = \deg s_{|C} = \deg \mathcal{O}_S(D)_{|C}$. By linearity, $(L \cdot \mathcal{O}_S(C)) = \deg L_{|C}$ for all $L \in \mathrm{Pic}(S)$.

Examples

① $S = \mathbb{P}^2$

$C \subset \mathbb{P}^2$ defined by a form $F_d(X,Y,Z)$ of degree d. $\dfrac{F_d}{Z^d}$ rational function $\Rightarrow$ $C \equiv dH$, H line in $\mathbb{P}^2$. Thus $\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}[H]$, $(C \cdot D) : \deg(C)\deg(D)$ (**Bézout theorem**).

② $S = \mathbb{P}^1 \times \mathbb{P}^1$

Put $A = \mathbb{P}^1 \times \{0\}$, $B = \{0\} \times \mathbb{P}^1$, $U = S \smallsetminus (A \cup B) \cong \mathbb{A}^2$.

$D \in \mathrm{Div}(S)$: $D_{|U} = \mathrm{div}(\varphi)$ for some rational function φ.

$D - \mathrm{div}\,\varphi = aA + bB$ for some $a, b \in \mathbb{Z} \implies$

$\mathrm{Pic}(\mathbb{P}^1 \times \mathbb{P}^1) = \mathbb{Z}[A] \oplus \mathbb{Z}[B]$. $(A \cdot B) = 1$ (transverse).

$A^2 = \big(A \cdot (\mathbb{P}^1 \times \{1\})\big) = 0$, $B^2 = 0$: intersection form $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Examples (continued)

③ $p : S \to C$, $F := p^{-1}(x)$. $\exists\, D \in \mathrm{Div}(C)$, $x \notin D$, $x \equiv D$; then $F \equiv p^*D \;\Rightarrow\; F^2 = F \cdot p^*D = 0$.

④ $D \geqslant 0$, $D \cdot C < 0 \;\Rightarrow\; D = C + E$, $E \geqslant 0$.

(otherwise $D = \sum n_i C_i$, $C_i \neq C \;\Rightarrow\; C \cdot C_i \geqslant 0 \;\forall\, i$)

⑤ $C^2 < 0, C \equiv D \geqslant 0 \;\Rightarrow\; D = C$ $(\Leftrightarrow h^0(\mathcal{O}_S(C)) = 1)$.

Canonical line bundle and Riemann-Roch

Ω^1_S = sheaf of differential 1-forms, locally isomorphic to $\mathcal{O}_S^2$
(locally $a(x,y)dx + b(x,y)dy$).

$\mathcal{K}_S = \bigwedge^2 \Omega^1_S$ = sheaf of 2-forms = **canonical line bundle**
(locally $\omega = f(x,y)dx \wedge dy, \mathrm{div}(\omega) = \mathrm{div}(f)$).

K_S or K = **canonical divisor** = divisor of any rational 2-form.

Example : $K_{\mathbb{P}^2} \equiv -3H$.

Indeed the 2-form $\dfrac{XdY \wedge dZ + YdZ \wedge dX + ZdX \wedge dY}{XYZ}$ is well-defined, does not vanish, and has a pole $\equiv 3H$.

Example : C_1, C_2 smooth projective curves, $S = C_1 \times C_2$, projections $p_i : S \to C_i$. Then $K_S \equiv p_1^* K_{C_1} + p_2^* K_{C_2}$.

Indeed if α_i is a 1-form on C_i (possibly rational), $p_1^*\alpha_1 \wedge p_2^*\alpha_2$ is a 2-form on S, with divisor $p_1^* \mathrm{div}(\alpha_1) + p_2^* \mathrm{div}(\alpha_2)$.

Riemann-Roch

Recall: $L \in \mathrm{Pic}(S) \rightsquigarrow H^i(S, L) = H^i(L)$, $i = 0, 1, 2$.

$h^i(L) = \dim H^i(L)$. $\chi(L) := h^0(L) - h^1(L) + h^2(L)$.

If $L = \mathcal{O}_S(D)$, we write $H^i(D)$, $h^i(D)$, $\chi(D)$.

Theorem

- **Riemann-Roch :** $\chi(L) = \chi(\mathcal{O}_S) + \frac{1}{2}(L^2 - \mathcal{K}_S \cdot L)$.
- **Serre duality :** $h^i(L) = h^{2-i}(\mathcal{K}_S \otimes L^{-1})$.

Since the term h^1 is difficult to control, we will most often use R-R as an inequality, using Serre duality. In divisor form:

$$h^0(D) + h^0(K - D) \geqslant \chi(\mathcal{O}_S) + \frac{1}{2}(D^2 - K \cdot D)\,.$$

The genus formula

Corollary (genus formula)

$$C \textit{ irreducible} \subset S \;\Rightarrow\; g(C) := h^1(\mathcal{O}_C) = 1 + \frac{1}{2}(C^2 + K\cdot C).$$

Proof : Exact sequence $0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0 \implies$

$$\chi(\mathcal{O}_C) = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) \overset{\text{R-R}}{=\!=} -\frac{1}{2}(C^2 + K\cdot C)\,. \quad \blacksquare$$

Examples : • $C \subset \mathbb{P}^2$ of degree $d \;\Rightarrow$

$$g(C) = 1 + \frac{1}{2}(d^2 - 3d) = \frac{1}{2}(d-1)(d-2)\,.$$

• $C \subset \mathbb{P}^1 \times \mathbb{P}^1$ of bidegree (p,q) (i.e. $C \equiv pA + qB$) $\Rightarrow$

$$g(C) = 1 + \frac{1}{2}(2pq - 2p - 2q) = (p-1)(q-1)\,.$$

The genus of a singular curve

Remark : *Let $n : N \to C$ be the normalization of C. Then $g(C) \geqslant g(N)$, with equality iff C is smooth.*

Proof : Exact sequence $\qquad 0 \to \mathcal{O}_C \to n_*\mathcal{O}_N \to \mathcal{T} \to 0$

with $\mathcal{T}$ concentrated on the singular points of C.

Hence $H^i(\mathcal{T}) = 0$ for $i > 0$. Therefore $\chi(\mathcal{O}_C) = \chi(\mathcal{O}_N) - h^0(\mathcal{T})$, and $g(C) = g(N) + h^0(\mathcal{T}) \geqslant g(N)$, equality iff $C = N$ smooth. ■

Corollary

$C^2 + K \cdot C \geqslant -2$; equality $\Rightarrow$ $C \cong \mathbb{P}^1$.

Indeed $C^2 + K \cdot C = 2g(C) - 2 \geqslant 2g(N) - 2 \geqslant -2$. ■

Numerical invariants

Algebraic surfaces are distinguished by their numerical invariants:

- The most important: K^2, $\chi(\mathcal{O})$.

  Though we will not use this in the lectures, I want to mention:

Theorem

1. *(M. Noether)* $K^2 \geqslant 2\chi(\mathcal{O}) - 6$;
2. *(Miyaoka-Yau)* $K^2 \leqslant 9\chi(\mathcal{O})$.

The relation of $K^2/\chi(\mathcal{O})$ with the geometry of the surface is a long chapter of surface theory ("geography").

Refined invariants:

- $h^2(\mathcal{O}) = h^0(K)$ (Serre duality), the **geometric genus** p_g;
- $h^1(\mathcal{O}) = H^0(\Omega^1)$ (Hodge theory), the **irregularity** q;
- $h^0(nK)$ $(n \geqslant 1)$, the **plurigenera** P_n.