Algebraic surfaces

Lecture II: Rational and birational maps

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Blowing up

Proposition

$p \in S$. $\exists\, b : \hat{S} \to S$, unique up to isomorphism, such that

1. $b^{-1}(p) = E \cong \mathbb{P}^1$;
2. $b : S \smallsetminus E \xrightarrow{\sim} S \smallsetminus p$.

Sketch of proof: coordinates x, y in $U \ni p$

$\hat{U} \subset U \times \mathbb{P}^1 : xY - yX = 0$.

$b : \hat{U} \to U$ projection, satisfies ① and ②.

Then glue $S \smallsetminus p$ and $\hat{U}$ along $U \smallsetminus p$. ■

In $\hat{U}' \subset \hat{U} : \{X \neq 0\}$, $y = xt$ with $t = \dfrac{Y}{X}$:

(x, t) local coordinates, $b(x, t) = (x, tx)$,

E given by $x = 0$.

The strict transform

We say that E is the **exceptional curve** of the blowing up.

$E \xrightarrow{\sim} \mathbb{P}(T_p(S))$: $(X, Y) \in E \leftrightarrow$ tangent direction $xY - yX = 0$.

For $C \subset S$, **strict transform** $\hat{C} :=$ closure of $C \smallsetminus p$ in $\hat{S}$.

$\hat{C} \cap E = \{$tangent directions to C at $p\}$.

Lemma

$b^*C = \hat{C} + mE$ in $\mathrm{Div}(\hat{S})$, where $m := m_p(C)$.

Proof : Eqn. of C in U: $0 = f(x, y) = f_m(x, y) + f_{m+1}(x, y) + ..$

Choose (x, y) such that $f_m(x, 0) \neq 0$, i.e. C not tangent to $y = 0$.

$b^*f = f(x, tx) = x^m\big(f_m(1, t) + xf_{m+1}(1, t) + \dots\big)$, $f_m(1, 0) \neq 0$

$\Rightarrow$ multiplicity of E in $\mathrm{div}(b^*f) = m$. ∎

The Picard group of $\hat{S}$

Proposition

1. $\mathrm{Pic}(\hat{S}) = b^*\,\mathrm{Pic}(S) \overset{\perp}{\oplus} \mathbb{Z}[E]$, $(b^*C \cdot b^*D) = (C \cdot D)$, $E^2 = -1$.
2. $K_{\hat{S}} = b^*K_S + E$.
3. $b_2(\hat{S}) = b_2(S) + 1$.

Proof : • $\Gamma \subset \hat{S}$, $\Gamma \neq E \;\Rightarrow\; \Gamma =$ strict transform of $b(\Gamma) \subset S$ $\Rightarrow\; \Gamma = b^*b(\Gamma) - mE$.

• $\forall\, C \subset S$, $C \equiv A \not\ni p \;\Rightarrow\; (b^*C \cdot E) = 0$, $(b^*C \cdot b^*D) = (C \cdot D)$.

• Take $H \ni p$, $m_p(H) = 1$. Then $(\hat{H} \cdot E) = 1$; $b^*H = \hat{H} + E$, $(b^*H \cdot E) = 0 \;\Rightarrow\; E^2 = -1$.

• $b^*K_S = K_{\hat{S}} + kE \;\Rightarrow\; K_{\hat{S}} \cdot E + kE^2 = 0$. $K_{\hat{S}} \cdot E = -1$ (genus formula) $\Rightarrow\; k = 1$.

• The claim on b_2 follows from standard topological arguments. ∎

Rational maps

Corollary

$C \subset S$, strict transform $\hat{C} \subset \hat{S}$. Then $(K_{\hat{S}} \cdot \hat{C}) \geqslant (K_S \cdot C)$.

Proof : $(K_{\hat{S}} \cdot \hat{C}) = (b^* K_S + E) \cdot (b^* C - mE) = (K_S \cdot C) + m.$ ■

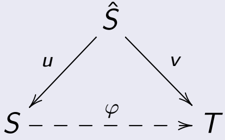
Definition : Rational map $\varphi : S \dashrightarrow T :=$ morphism $S \supset U \to T$. We'll always take the largest U such that $\varphi_{|U}$ is a morphism.

- φ is **birational** if $\exists\, U \subset S,\ V \subset T$ such that $\varphi : U \xrightarrow{\sim} V$
  - then we say that S and T are birational.

Elimination of indeterminacy

Theorem (Elimination of indeterminacy)

1. *$\exists\, u, v$ morphisms, $u = b_1 \circ \ldots \circ b_n$ blowups.*



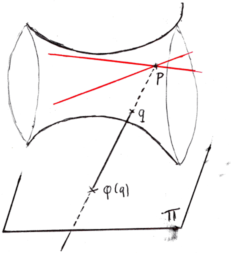
2. *A birational* morphism *is a composition of blowups.*

Remark : ① holds in higher dimension ("Hironaka's little roof"), but not ②.

Example: stereographic projection

$Q \subset \mathbb{P}^3$ smooth quadric $XT - YZ = 0$. Segre embedding
$s : \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sim} Q \subset \mathbb{P}^3$, $s(U,V;W,S) = (UW, US, VW, VS)$.

For each $p = s(a,b) \in Q$, there are 2 lines $\subset Q$ passing through p: $s(\mathbb{P}^1 \times b)$ and $s(a \times \mathbb{P}^1)$.



Let $\Pi \subset \mathbb{P}^3$ plane $\not\ni p$.

$\varphi : Q \dashrightarrow \Pi$: $q \neq p \rightsquigarrow \langle p, q\rangle \cap \Pi$.

Extension $f : \hat{Q} \to \Pi$: $\ell \in \mathbb{P}(T_p(Q)) \mapsto \ell \cap \Pi$.

f birational, contracts the 2 lines through p.

$$\begin{array}{ccc} & \hat{Q} & \\ {}^{b}\swarrow & & \searrow^{f} \\ \mathbb{P}^1 \times \mathbb{P}^1 = Q & \overset{\varphi}{\dashrightarrow} & \Pi = \mathbb{P}^2 \end{array}$$

Some consequences

Corollary

$\varphi : S \dashrightarrow T$ rational. $\exists\, F \subset S$ finite, $\varphi : S \smallsetminus F \to T$ morphism.

Remark : Direct proof easy, see exercises.

Consequences : • Since $\operatorname{Div}(S) \xrightarrow{\sim} \operatorname{Div}(S \smallsetminus F)$ and $\operatorname{Pic}(S) \xrightarrow{\sim} \operatorname{Pic}(S \smallsetminus F)$, $\varphi^* : \operatorname{Div}(T) \to \operatorname{Div}(S)$ and $\operatorname{Pic}(T) \to \operatorname{Pic}(S)$ defined.

- For $C \subset S$, $\varphi(C) := \overline{\varphi(C \smallsetminus F)}$ well-defined.
- $\varphi : S \overset{\sim}{\dashrightarrow} T \;\Rightarrow\; H^0(T, K_T) \xrightarrow{\sim} H^0(S, K_S)$.

  (Beware! Not true that $\varphi^* K_T = K_S$, think of blowups)

Proof : $\varphi^* : H^0(T, K_T) \to H^0(S \smallsetminus F, K_S) \xleftarrow{\sim} H^0(S, K_S)$, then $(\varphi^{-1})^* : H^0(T, K_T) \to H^0(S, K_S)$ inverse of φ^*. ■

- $H^0(T, nK_T) \xrightarrow{\sim} H^0(S, nK_S)$ for $n > 0$ (same argument).
- $H^0(T, \Omega^1_T) \xrightarrow{\sim} H^0(S, \Omega^1_S)$ (same argument).

Birational invariants

- The numerical invariants $p_g(S) := h^0(K_S)$ (**geometric genus**), $P_n(S) := h^0(nK_S)$ (**plurigenera**), $q(S) := h^0(\Omega^1_S)$ (**irregularity**) are **birational invariants**.

Definition

A surface is **ruled** if it is birational to $C \times \mathbb{P}^1$.

Proposition

S ruled $\Rightarrow\ P_n(S) = 0\ \forall n \geqslant 1$.

Proof : Suffices to prove it for $S = C \times \mathbb{P}^1$.

$F = \{c\} \times \mathbb{P}^1$ satisfies $F^2 = 0$, hence $K \cdot F = -2$ (genus formula).

If $nK \equiv D \geqslant 0$, D must contain $\{c\} \times \mathbb{P}^1$ for all $c \in C$, impossible. ■

Irregularity of ruled surfaces

The converse is true, but difficult:

Theorem (Enriques)

$P_n(S) = 0 \ \forall n \ \Rightarrow \ S$ *ruled.*

In fact Enriques proved a more precise result: $P_{12} = 0 \ \Rightarrow \ S$ ruled.

Proposition

S *birational to* $C \times \mathbb{P}^1 \ \Rightarrow \ q(S) = g(C)$.

Proof: $S = C \times \mathbb{P}^1 \xrightarrow{p} C$. **Claim:** $p^* : H^0(C, K_C) \xrightarrow{\sim} H^0(S, \Omega^1_S)$.
$\omega \in H^0(\Omega^1_S)$, $s : C \hookrightarrow C \times \mathbb{P}^1$, $s(c) = (c, 0)$. Suffices: $\omega = p^* s^* \omega$.
Local coordinates z on C, t on $\mathbb{P}^1 \ \rightsquigarrow \ \omega = a(z,t)dz + b(z,t)dt$.
$\omega_{\{c\} \times \mathbb{P}^1} = 0 \ \Rightarrow \ b(c,t) \equiv 0 \ \forall c \ \Rightarrow \ b = 0$.
$d\omega \in H^0(K_S) = 0 \ \Rightarrow \ \dfrac{\partial}{\partial t} a(z,t) = 0 \ \Rightarrow \ a(z,t) = a(z,0)$,
$\omega = a(z,0)dz = p^* s^* \omega$. ∎

Minimal surfaces

Definition

S **minimal** if any birational morphism $S \to T$ is an isomorphism.

Proposition

Every S admits a birational morphism onto a minimal surface.

Proof : If not, $\exists$ an infinite chain $S \to S_1 \to \cdots \to S_n \to \cdots$ of blowups. This is impossible since $b_2(S_n) = b_2(S) - n$. ∎

Theorem (Castelnuovo's criterion)

Let $E \subset S$, $E \cong \mathbb{P}^1$, $E^2 = -1$. There exists a surface T and a blowing up $b : S \to T$ with exceptional curve E.

Corollary

S minimal $\Leftrightarrow$ $S \not\ni E \cong \mathbb{P}^1$ with $E^2 = -1$.

Non-ruled surfaces

Theorem

Two birational minimal surfaces **not** *ruled are isomorphic.*

Thus a non-ruled surface admits a **unique** minimal model (up to isomorphism); the birational classification of these surfaces is reduced to the classification (up to isomorphism) of the minimal ones. In contrast, ruled surfaces have a simple birational model ($C \times \mathbb{P}^1$), but the determination of the minimal ones is subtle.

The theorem follows easily from an important Lemma (admitted):

Key lemma

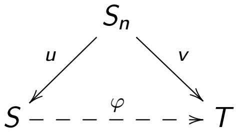
If S is minimal not ruled, $(K \cdot C) \geqslant 0$ for all curves C.

We say that K is **nef**. This is the crucial notion to extend the definition of minimal surface in higher dimension.

Proof of the Theorem

Let $\varphi : S \overset{\sim}{\dashrightarrow} T$, with S, T minimal not ruled. We want to prove that φ is an isomorphism.

We choose a diagram:



v birational, $u : S_n \to S_{n-1} \to \cdots \to S_0 = S$, with $n \geqslant 1$ minimal $\Rightarrow$ v maps E_n to a curve C.

Since v is a composition of blowups,

$(K_T \cdot C) \leqslant (K_{S_n} \cdot E_n) = -1$, contradicting the key lemma. ■