2020 AG Summer School

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References for Lecture I-III: Hartshorne "Algebraic Geometry" Chapter I and Chapter II; Vakil "Foundations of Algebraic Geometry" Part II and Part V

1 Projective Scheme

All the rings will be commutative.

1.1 A warm up for projective geometry

* **Projective spaces as complex manifolds**

  $\mathbb{P}^n_{\mathbb{C}} := \mathbb{C}^n - \{(0,\ldots,0)\}/\sim$ has a complex manifold structure by associating a holomorphic local charts.

  Definition 1.1.1 (projective complex manifold)**.** A complex manifold M is said to be projective if there is a closed embedding $M \hookrightarrow \mathbb{P}^n_{\mathbb{C}}$ for some n.

  Typical examples (from last week)

  - The compact Riemman surfaces of genus g
  - The Grassmannian $\mathrm{Gr}(r,n)$ can be embedded into $\mathbb{P}^{\binom{n}{k}-1}$ via the Plücker embedding.
  - product, $\mathbb{P}^n$-bundle, polarized families over projective objects

* **Projective spaces as schmems/varieties**

  We have seen from last week: $\mathbb{P}^n_{\mathbb{C}}$ can be viewed as a scheme obtained by gluing $n+1$ open subsets

$$U_i \cong \mathbb{A}^n_{\mathbb{C}} = \mathrm{Spec}\left(\mathbb{C}\left[\frac{X_0}{X_i},\ldots,\frac{X_n}{X_i}\right]\right)$$

  along the overlaps $U_{ij} = U_i \cap U_j$ via the transistion function $\frac{X_j}{X_i} \mapsto \frac{X_i}{X_j}$.

  The projective schemes over $\mathbb{C}$ are closed subschemes of $\mathbb{P}^n_{\mathbb{C}}$ under Zariski topology. The projective complex varieties are obtained by taking the closed points of projective schemes. Such varieties can be viewed as the solution set of homogenous polynomial equations

$$f_1(x_0, \dots, x_n) = \dots = f_m(x_0, \dots, x_n) = 0.$$

where f_i are homogenous.

- **Chow's Theorem/GAGA by Serre**: there is an equivalence

$$\{\texttt{projective complex manifolds }\} \Leftrightarrow \{\texttt{Projective complex varieites}\}$$

Goal of today: *functorial algebraic constructions of projective objects*

1.2 Proj constructions

The Spectrum functor defines an equivalence

$$\text{Spec} : \{\texttt{rings}\} \longrightarrow \{\texttt{affine schemes}\}$$
$$R \mapsto \text{Spec } R$$

The projective schemes can be obtained via so called Proj functor.

Definition 1.2.1 (**Proj** construction for graded ring)**.**
Let $S = \bigoplus_{d \geq 0} S_d$ be a graded ring and $S_+ = \bigoplus_{d>0} S_d$ the irrelevant ideal. Then

$$\text{Proj } S = \{\mathfrak{p} \in \text{Spec } S | \ \mathfrak{p} \text{ is homogenous}, S_+ \not\subset \mathfrak{p}\}$$

We endow it with the induced topology.

- $\forall f \in S$ homogenous of degree d, there is a standard open subset

  $$D_+(f) = \{\mathfrak{p} \in \text{Proj } S | f \notin \mathfrak{p}\} \cong \text{Spec } S_{(f)}$$

  where $S_{(f)}$ is the subring of S_f consisting of elements of the form r/f^n with r homogeneous and $\deg(r) = nd$.

- The structure sheaf $\mathcal{O}_{\text{Proj } S}$ on Proj S is the unique sheaf of rings $\mathcal{O}_{\text{Proj } S}$ which agrees with $\mathcal{O}_{\text{Spec } (S_{(f)}}$ on the standard open subset $D_+(f)$.

Example 1.
1. When $S = k[x_0, \dots, x_n] = \bigoplus S_d$ with the usual grading, then Proj $S = \mathbb{P}^n_k$.
2. Write $T = k[y_0, \dots, y_m] = \bigoplus T_d$. Then

$$\text{Proj } (\bigoplus_d S_d \otimes T_d) = \mathbb{P}^n_k \times \mathbb{P}^m_k.$$

FACT: Proj defines a functor

$$\text{Proj} : \{\texttt{graded ring over A}\} \to \{\texttt{projective scheme over A}\}$$

Some examples for morphisms between projective varieties.

Example 2.

(1) Veronese or d-uple embedding: $\varphi_d : \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$ sending $[x_0, \ldots, x_n]$ to $[x_0^d, x_0^{d-1}x_1, \ldots, x_n^d]$.

(2) Segre embedding: $\mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$ sending $([x_0, \ldots, x_n], [y_0, \ldots, y_m])$ to $[x_0y_0, \ldots, x_ny_m]$.

Note that (2) also implies that the product of projective varieties over k remains projective.

The construction of Proj of a graded sheaf gives rise to a projective morphism.

Definition 1.2.2 (Proj construction of graded sheaf)**.**
• A graded quasicoherent sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules means

$$\mathcal{F} = \bigoplus_{d \geq 0} \mathcal{F}_d$$

satisfying $\mathcal{F}_d \cdot \mathcal{F}_{d'} \subseteq \mathcal{F}_{d+d'}$ and $\mathcal{F}_0 = \mathcal{O}_X$.
• We can define Proj $\mathcal{F}$ by gluing the scheme Proj $\mathcal{F}(U)$, $U \subseteq X$.

Example 3. If $\mathcal{E}$ is a locally free sheaf on X, then $\mathrm{Sym}^{\bullet}\mathcal{E} = \bigoplus \mathrm{Sym}^d \mathcal{E}$ is a graded $\mathcal{O}_X$-module. We obtain a projective bundle

$$\mathbb{P}(\mathcal{E}) = \mathrm{Proj}\ (\mathrm{Sym}^{\bullet}\mathcal{E})$$

over X.

Basic properties of a projective scheme.

(a) Let X be a projective variety over k. Then X is proper and $H^0(X, \mathcal{O}_X) = k$.

The converse is almost true:

Chow's Lemma: *Every proper variety is birational to a projective variety.*

(b) (Twisted sheaf) Suppose $S = k[x_0, \ldots, x_n]$ is generated by S_1. The projective scheme Proj S carries a natural invertible sheaf $\mathcal{O}_S(1) := \widetilde{S}(1)$.

E.g. the projective space $\mathbb{P}_k^n$ carries a natural invertible sheaf $\mathcal{O}_{\mathbb{P}_k^n}(1)$. Hence the projective subvariety $X \subseteq \mathbb{P}_k^n$ can be endowed with an invertible sheaf $\mathcal{O}_X(1)$ via restriction.