2 Geometry of projective varieties

Classical problem: find $\sharp$ of polynomial equations. Geometrically, this is related to how varieties intersects.

The answer of this problem is to relate $\sharp$ to some invariants of projective varieties.

2.1 Invariants of projective varieties

A motivating example is

Example 4 (Gauss' fundamental theorem of algebra)**.** The polynomial equation $f(z) = 0$ has $\deg(f)$ solutions (with multiplicity) in $\mathbb{C}$. Equivalently, the homogenous polynomial equation $f(x, y) = 0$ has $\deg(f)$ solutions.

For three variables, the fundamental result is the following:

Theorem 2.1.1 (Bézout theorem for plane curves)**.** *Let f, g be two distinct irreducible homogenous polynomials in $k[x, y, z]$. The equations*

$$f(x, y, z) = g(x, y, z) = 0$$

have $\deg(f) \cdot \deg(g)$ *solutions (with multiplicity).*

In other words, the two plane curves $C_1 = \{f(x, y, z) = 0\}$ and $C_2 = \{g(x, y, z) = 0\}$ in $\mathbb{P}^2$ meet at $\deg(f)\deg(g)$ points.

Remark. The Bézout's theorem tells that any two closed curves in $\mathbb{P}^2$ will have non-empty intersections. Note this fails for affine varieties, i.e. two affine lines in $\mathbb{A}^2$ do not necessarily meet.

The higher dimensional generalization requires the concept of Hilbert polynomial.

* Hilbert polynomial of projective varieties

Definition 2.1.2. Let $\mathcal{F}$ be a coherent sheaf on a projective scheme $X \subseteq \mathbb{P}^n$. By **Hilbert-Serre**, there eixsts a polynomial $P_{\mathcal{F}}(z) \in \mathbb{Q}[z]$ such that

$$P_{\mathcal{F}}(d) = \chi(\mathcal{F}(d)) = \sum_{i \geq 0} (-1)^i h^i(X, \mathcal{F}(d))$$

for $d >> 1$, where $\mathcal{F}(d) = \mathcal{F} \otimes \mathcal{O}_X(d)$. $P_{\mathcal{F}}(z)$ is the Hilbert polynomial of $\mathcal{F}$ and $P_X := P_{\mathcal{O}_X}$ is called the Hilbert polynomial of X in $\mathbb{P}^n$.

Facts for P_X (not very trivial)

1. $P_X(d) = h^0(X, \mathcal{O}_X(d))$ for d sufficiently large due to the Serre vanishing theorem, i.e. $H^i(X, \mathcal{F}(d)) = 0$ for $i > 0$ if $\mathcal{F}$ is coherent and d sufficiently large.
2. First invariant: $\deg(P_X) = \dim X = m$.

3. Seconding invariant: leading coefficients of P_X is $\frac{\deg(X)}{m!}$.
4. Invariant from the constant term: the arithmetic genus of X: $(-1)^m(P_X(0)-1)$.

All these invariants are deformation invariant.

Example 5 (Invariants determines the geometry)**.** 1. if $\deg(X) = 1$, then X is a projective linear subspace in $\mathbb{P}^n$.

2. More generally, if X is non-degenerate in $\mathbb{P}^n$, then $\dim X + \deg(X) \geq n+1$.

∗ Bézout's theorem

With the knowledge of the degree, we can state Bézout's theorem in arbitrary dimensional projective space.

Theorem 2.1.3. *Let X be a projective variety in $\mathbb{P}^n_k$ with $\dim X \geq 1$ and H be a hypersurface not containing X. Denote by Z_i the irreducible components of the intersection of H and X. Then*

$$\deg(X) \cdot \deg(H) = \sum_{Z_i} \mu(X, H; Z_i) \deg(Z_i)$$

where $\mu(X, H; Z_i)$ is the intersection multiplicity at Z_i.

The proof relies on computing the Hilbert polynomials via the short exact sequence

$$0 \to \mathcal{O}_X(-\deg(H)) \to \mathcal{O}_X \to \mathcal{O}_{X\cap H} \to 0$$

Remark: if $\mathfrak{p}_i$ is the prime ideal corresponds to Z_i, then $\mu(X, H; Z_i)$ is the length of $(k[x_0, \ldots, x_n]/(I_X + I_H))_{\mathfrak{p}_i}$ as a $k[x_0, \ldots, x_n]_{\mathfrak{p}_i}$-module.

Important consequences

- For projective variety X in $\mathbb{P}^n$ of dimension d, the intersection number with d general hyperplanes

$$H_1 \cdot H_2 \ldots \cdot H_d \cdot X$$

  is positive. As all hyperplanes are linearly equivalent, it is the same as $H^d \cdot X > 0$.
- More generally, if we call the intersection $L := X \cap H$ the hyperplane class on Y, then $L^d \cdot Y > 0$ for any subvariety $Y \subseteq X$ of dimension d.

2.2 Generic intersection

Among the questions for intersection multiplicity, a natural one is when the intersection multipicity will be one.

Definition 2.2.1. Let X be a variety over k. A point $p \in X$ is smooth $\dim X = \dim T_pX$.

Theorem 2.2.2 (Bertini Theorem)**.** *Let $X \subseteq \mathbb{P}(V) \cong \mathbb{P}^n$ be a smooth subvariety of dimension greater than zero. Then for a generic hypersurface H, $Y = X \cap H$ is again smooth.*

Proof. 1. Note that the set of hyperplanes is parametrized by the dual projective space $\mathbb{P}(V^\vee)$.

2. To say that a hyperplane is generic is equivalent to saying that there is a nonempty open subset $U \subseteq \mathbb{P}(V^\vee)$ consisting of points corresponding to that hyperplane and such that each hyperplane in U possesses the desired property.

3. $H \cap X$ will be smooth at x if $T_xX \not\subset T_xH$.

4. Consider the subset

$$Z = \{(H, x) \mid x \in H, T_xX \subset T_xH\} \subseteq \mathbb{P}(V^\vee) \times X,$$

it is a closed subset.

5. The set of H in $\mathbb{P}(V^\vee)$ for which $H \cap X$ is singular is the image of Z via the projection $\mathbb{P}(V^\vee) \times X \to \mathbb{P}(V^\vee)$.

6. The assertion follows by an easy dimension count: $dim(Z) = n - 1$. □

A more general statement is as follows:

Theorem 2.2.3. *Suppose* $\mathrm{char}(k) = 0$. *Then for any linear system $f : X \dashrightarrow \mathbb{P}^n_k$ and H a generic hyperplane, the pullback $f^{-1}(H)$ is smooth outside the base locus of f.*

It fails in positive characteristic fields because the existence of purely inseparable map.