Algebraic surfaces

Lecture IV: Rational surfaces

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Linear systems and rational maps

$L = \mathcal{O}_S(D) \in \mathrm{Pic}(S)$. (Complete) **linear system :**

$$|L| = |D| := \{E \geqslant 0 \mid E \equiv D\} = \mathbb{P}(H^0(L)).$$

B_L = **Base locus** of $L := \bigcap_{E \in |L|} E = Z \bigcup \{p_1, \ldots, p_s\}$

$Z = \bigcup C_i$ = **fixed part**, p_i **base points**.

Rational map defined by L:

$\varphi_L : S \smallsetminus B_L \to |L|^\vee$, $\varphi_L(p) = \{E \mid p \in E\}$ = hyperplane in $|L|$.

If Z = fixed part of $|L|$, $\varphi_L = \varphi_{L(-Z)}$: can assume L has no fixed part, i.e. B_L finite.

$E \in |L| \rightsquigarrow$ hyperplane $H_E \subset |L|^\vee$;

$\varphi_L^* H_E = \{p \in S \smallsetminus B_L \mid E \in \varphi_L(p) \Leftrightarrow p \in E\} = E \smallsetminus B_L$: $\varphi_L^* H_E = E$.

Properties of φ_L

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- φ_L morphism $\Leftrightarrow |L|$ base point free (i.e. $B_L = \varnothing$).
- φ_L injective $\Leftrightarrow \forall\, p \neq q,\ \exists\, E \in |L|,\ p \in E,\ q \notin E$.
- φ_L embedding $\Leftrightarrow \forall\, p, v \neq 0 \in T_p(S),\ \exists\, p \in E \in |L|,\ v \notin T_p(E)$.

  If this is the case, we say that L is **very ample**.
- φ_L embedding $\Rightarrow \deg(\varphi_L(S)) = L^2$.

Remark : If D is very ample and $|E|$ is base point free, $D + E$ is very ample.

Examples : • Let H be a line in $\mathbb{P}^2$. The linear system $|nH|$ of curves of degree n $(n \geqslant 1)$ is very ample. In particular, φ_{2H} is an isomorphism of $\mathbb{P}^2$ onto a surface $V \subset \mathbb{P}^5$, the **Veronese surface**.

Examples

- On $\mathbb{P}^1 \times \mathbb{P}^1$, let $A = \mathbb{P}^1 \times \{0\}$ and $B = \{0\} \times \mathbb{P}^1$. The linear systems $|A|$ and $|B|$ are base point free, and φ_{A+B} is the Segre embedding in $\mathbb{P}^3$. Hence $aA + bB$ is very ample for $a, b \geqslant 1$. In particular, $|2A + B|$ gives an isomorphism onto a surface of degree 4 in $\mathbb{P}^5$ ("quartic scroll"). Since $A \cdot (2A + B) = 1$, the curves in $|A|$ are mapped to lines in $\mathbb{P}^5$.

- Let $p_1, \ldots, p_s \in S$. Let $|D|$ be a linear system on S, and $P \subset |D|$ the subspace of divisors passing through $p_1, \ldots, p_s$. Assume that at each p_i the curves of P have different tangent directions. Let $b : \hat{S} \to S$ be the blowing up of $p_1, \ldots, p_s$, E_i the exceptional curve above p_i. The system $\hat{D} := b^*D - \sum E_i$ is base point free and defines a morphism $\varphi_{\hat{D}} : \hat{S} \to |\hat{D}|^\vee$ to which we can apply the previous remarks.

Examples (continued)

- Let $p \in \mathbb{P}^2$; consider the system of conics passing through p. It is easy to check that the system $2b^*H - E$ on $\hat{\mathbb{P}}^2_p$ is very ample. It gives an isomorphism onto a surface $S \subset \mathbb{P}^4$. We have $\deg(S) = (4H^2 + E^2) = 3$. The strict transforms of the lines through p in $\mathbb{P}^2$ form the linear system $b^*H - E$; since $(b^*H - E) \cdot (2b^*H - E) = 1$, they are mapped to lines in $\mathbb{P}^4$. S is the **cubic scroll**.

- Now let us pass to linear systems of cubic curves.

Proposition

For $s \leqslant 6$, let $p_1, \ldots, p_s \in S = \mathbb{P}^2$, such that no 3 of them lie on a line and no 6 on a conic. The linear system $|-K|$ on $\hat{S}$ is very ample, and defines an isomorphism of $\hat{S}$ onto a surface Σ_d of degree $d := 9 - s$ in $\mathbb{P}^d$, called a ***del Pezzo surface****.*

Sketch of proof

Sketch of proof : The proof is a long exercise, with no essential difficulty; I will just give an idea. We have $-K_{\hat{S}} = 3b^*H - \sum E_i$, corresponding to the system P of cubics passing through the p_i. Let us show that φ_{-K} is injective in the most difficult case $s = 6$.

- Let $p \neq q \in \mathbb{P}^2 \smallsetminus \{p_i\}$. Can assume p_1 is not on the line $\langle p, q\rangle$.
- $\exists\,!$ conic Q_{ij} passing through p and the p_k for $k \neq i, j$.
- $Q_{1i} \cap Q_{1j} = \{p\} \cup 3$ other $p_k \Rightarrow q \in$ at most one Q_{1i}, say Q_{12}.
- q is at most on one $\langle p_1, p_i\rangle$, say $\langle p_1, p_3\rangle$.
- Then $Q_{14} \cup \langle p_1, p_4\rangle \in P$, $\ni p$, $\not\ni q \Rightarrow \varphi_{-K}(p) \neq \varphi_{-K}(q)$.
- Then: $\deg(\Sigma_d) = (3b^*H - \sum E_i)^2 = 9 - s = d$; one has $h^0(3H) = 10$, and one checks that $p_1, \ldots, p_s$ impose s independent conditions. ∎

Example : Σ_3 is a smooth cubic surface in $\mathbb{P}^3$; we will see that one obtains all smooth cubic surfaces in that way.

Lines on del Pezzo surfaces

Proposition

lines $\subset \Sigma_d =$ *exceptional curves* $=$ *the* E_i*, the strict transforms of the lines* $\langle p_i, p_j \rangle$ *and of the conics passing through* 5 *of the* p_i (*for* $s = 5$ *or* 6). *Their number is* $s + \binom{s}{2} + \binom{s}{5}$.

Proof : $E \subset \hat{S} \rightsquigarrow$ line in $\Sigma \Leftrightarrow K_{\hat{S}} \cdot E$=-1, i.e. E exceptional.
$E \neq E_i \Rightarrow E \equiv mb^*H - \sum a_i E_i$ in $\mathrm{Pic}(\hat{S})$; $a_i = E \cdot E_i = 0$ or 1.
$(-K) \cdot E = 3m - \sum a_i = 1 \Rightarrow \sum a_i = 2$ and $m = 1$, or $\sum a_i = 5$ and $m = 2$. ∎

Remark : We know more than the number of lines, namely their classes in $\mathrm{Pic}(\Sigma_d)$, their incidence properties, etc. The configuration of lines has been intensively studied in the 19th and 20th century. Let us just mention that the lattice $K^\perp \subset \mathrm{Pic}(\Sigma_d)$ is a *root system*, of type E_6, D_5, A_4, $A_2 \times A_1$ for $s = 6, 5, 4, 3$.

The cubic surface

Proposition

Any smooth cubic surface $S \subset \mathbb{P}^3$ is a del Pezzo surface Σ_3.

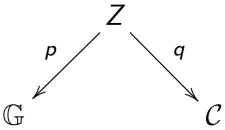
In particular, S contains 27 lines.

Strategy of the proof : show that S contains a line, then 2 skew lines; then deduce from that a map $S \to \mathbb{P}^2$ composite of blowups.

① $\mathbb{G} :=\{\text{lines} \subset \mathbb{P}^3\}$, $\dim \mathbb{G} = 4$.

$\mathcal{C} := |\mathcal{O}_{\mathbb{P}^3}(3)| = \{\text{cubic surfaces} \subset \mathbb{P}^3\} \cong \mathbb{P}^c$ $(c = 19)$.

Incidence correspondence: $Z \subset \mathbb{G} \times \mathcal{C} = \{(\ell, S) \mid \ell \subset S\}$.



Fibers of $p \cong \mathbb{P}^{c-4}$ ($S : F = 0$ contains $Z = T = 0 \Leftrightarrow F$ has no X^3, X^2Y, XY^2, Y^3).

Thus $\dim Z = \dim \mathcal{C}$. We want q surjective.

Cubic surface (continued)

If $q : Z \to \mathcal{C}$ not surjective, $\dim q(Z) \leqslant c-1 \;\Rightarrow\; \dim q^{-1}(S) \geqslant 1$ for $S \in q(Z)$. But $q^{-1}(\Sigma_3)$ finite $\Rightarrow$ impossible.

② $S \supset \ell$. The planes $\Pi \supset \ell$ cut S along a conic.

Claim : 5 of these conics are degenerate, i.e. of the form $\ell_1 \cup \ell_2$.

Proof : $\ell :\; Z = T = 0 \;\Rightarrow$

$F = AX^2 + 2BXY + CY^2 + 2DX + 2EY + G$, with $A, \ldots, G$ homogeneous polynomials in Z, T. The conic is degenerate

$$\Leftrightarrow \det \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & G \end{vmatrix} = 0,$$ degree 5 in Z, T. $\geqslant 2$ distinct roots $\Rightarrow$

$S \supset 2$ triangles: $\ell \cup \ell_1 \cup \ell_1'$, $\ell \cup \ell_2 \cup \ell_2'$. Then $\ell_1 \cap \ell_2 = \varnothing$.

Cubic surface (continued)

③ $\ell \subset S$, given by $X = Y = 0$. Projection from ℓ: $S \xrightarrow{(X,Y)} \mathbb{P}^1$.

Well-defined: $S : XB - YA = 0$, $(X, Y) = (A, B)$ on S,

$X = Y = A = B = 0 \;\Rightarrow\; S$ singular.

$\varphi_i : S \to \mathbb{P}^1$ projection from $\ell_i \;\rightsquigarrow\; \varphi = (\varphi_1, \varphi_2) : S \to \mathbb{P}^1 \times \mathbb{P}^1$.

Geometrically, $\varphi_i(p) =$ plane $\langle \ell_i, p \rangle$ through ℓ_i.

Birational: for $(\pi_1, \pi_2) \in \mathbb{P}^1 \times \mathbb{P}^1$, $\pi_1 \cap \pi_2 =$ line meeting ℓ_1 and ℓ_2, intersects S along a unique third point p.

Thus $\varphi =$ composition of blowups. Since blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ at 1 point $=$ blowup of $\mathbb{P}^2$ at 2 points, get $\varphi' : S \to \mathbb{P}^2$ composition of blowups. ■