Algebraic surfaces

Lecture V: The Kodaira dimension

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Kodaira dimension

The key ingredient to distinguish different projective varieties is the behaviour of the canonical bundle.

Definition

The **Kodaira dimension** of a surface S is

$$\kappa(S) := \max_n \dim \varphi_{nK}(S)$$

with the convention $\dim \varnothing = -\infty$.

Using the plurigenera $P_n = h^0(nK)$, this translates as

- $\kappa(S) = -\infty \iff P_n = 0 \ \forall n \iff S$ ruled (Enriques theorem).
- $\kappa(S) = 0 \iff P_n = 0$ or $1 \ \forall n$, and $= 1$ for some n.
- $\kappa(S) = 1 \iff P_n \geqslant 2$ for some n, and $\dim \varphi_{mK}(S) \leqslant 1 \ \forall m$;
- $\kappa(S) = 2 \iff \dim \varphi_{nK}(S) = 2$ for some n.

Examples

- Let B, C be two curves of genus b, c. Then:
  - $\kappa(B \times C) = -\infty \;\Leftrightarrow\; bc = 0$;
  - $\kappa(B \times C) = 0 \;\Leftrightarrow\; b = c = 1$;
  - $\kappa(B \times C) = 1 \;\Leftrightarrow\; b$ or $c = 1$, $bc > 1$;
  - $\kappa(B \times C) = 2 \;\Leftrightarrow\; b$ and $c \geqslant 2$.
- Let $S_d \subset \mathbb{P}^3$ of degree d; then S_d is rational for $d \leqslant 3$, $\kappa(S_4) = 0$, $\kappa(S_d) = 2$ for $d \geqslant 5$.

These examples show a general pattern: most surfaces have $\kappa = 2$ (they are called **of general type**), some have $\kappa = 1$, and the cases $\kappa = 0$ and $\kappa = -\infty$ are completely classified.

$\kappa = 2$

Proposition

Let S be a minimal surface. The following are equivalent:

1. $\kappa(S) = 2$;
2. $K^2 > 0$ *and S not rational;*
3. φ_{nK} *birational onto its image for* $n \gg 0$.

Proof : ③ $\Rightarrow$ ① clear.

② $\Rightarrow$ ③: let H be a very ample divisor on S. Riemann-Roch $\rightsquigarrow$
$\chi(nK - H) \sim \frac{1}{2} n^2 K^2 > 0$ for $n \gg 0$, hence
$h^0(nK - H) + h^0((1-n)K + H) > 0$.
But $\big((1-n)K + H\big) \cdot K < 0$ for $n \gg 0$, hence $h^0 = 0$ by key Lemma
$\Rightarrow$ $h^0(nK - H) > 0$, hence $nK \equiv H + E$, $E \geqslant 0 \Rightarrow \varphi_{nK}$ birational.

$\kappa = 2$ (continued)

① $\Rightarrow$ ②: Follows from:

Lemma

S minimal, $K^2 = 0$, $|nK| = Z + M$ with Z fixed part. Then M is base-point free, and $\varphi_M = \varphi_{nK} : S \to C \subset |nK|^\vee$.

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Proof : Key lemma $\Rightarrow$ $(K \cdot Z)$ and $(K \cdot M) \geqslant 0$, hence $= 0$.

$0 = M \cdot (Z + M) \ \Rightarrow\ M^2 = 0 \ \Rightarrow\ |M|$ base-point free, hence

$\varphi_M : S \to C \subset |nK|^\vee$. $M^2 = 0 \ \Rightarrow\ C$ curve. ■

Remark: $\exists$ much more precise results for ③ (Kodaira, Bombieri): φ_{nK} morphism for $n \geqslant 4$, birational for $n \geqslant 5$.

Example: For $S = B \times C$ as above,

$K^2_{B\times C} = (p^*K_B \cdot q^*K_C) = (2b-2)(2c-2)$: $K^2_X > 0 \Leftrightarrow b, c \geqslant 2$.

Surfaces with $\kappa = 1$

Proposition

S minimal, $\kappa(S) = 1 \;\Rightarrow\; K^2 = 0$, and $\exists\, p : S \to B$ with general fiber elliptic curve.

(We say that S is an **elliptic surface**.)

Proof : Choose n such that $h^0(nK) \geqslant 2$, $|nK| = Z + |M|$. By the Lemma, $\varphi_M : S \to C$.

Stein factorization: $\varphi_M : S \xrightarrow{p} B \to C$, with fibers of p connected.

F smooth fiber. $F \leqslant M \;\Rightarrow\; K \cdot F = 0,\ F^2 = 0 \;\Rightarrow\; g(F) = 1$ (genus formula). ■

Remark : An elliptic surface can be rational, ruled, or have $\kappa = 0$.

Surfaces with $\kappa = 0$

Theorem

S minimal with $\kappa = 0$.

1. *$q = 0$, $K \equiv 0$: S is a* **K3 surface***;*
2. *$q = 0$, $2K \equiv 0$, $K \not\equiv 0$: S is an* **Enriques surface** *– quotient of a K3 by a fixed-point free involution.*
3. *$q = 1$: S is a* **bielliptic surface***, quotient of a product $E \times F$ of elliptic curves by a finite group acting freely (7 cases).*
4. *$q = 2$: S is an* **abelian surface** *(projective complex torus).*

We will treat only the cases with $q = 0$ (the other cases require the theory of the Albanese variety). If $K \equiv 0$, we are in case ①.

We want to prove that $q = 0,\ K \not\equiv 0 \ \Rightarrow\ 2K \equiv 0$.

S minimal, $q = 0$, $K \not\equiv 0$

Proof : For some n, $P_n \geqslant 1$; by the key Lemma $K^2 \geqslant 0$, and $K^2 = 0$ by the case $\kappa = 2$.

By Riemann-Roch, $h^0(2K) + h^0(-K) \geqslant \chi(\mathcal{O}_S) \geqslant 1$.

If $h^0(-K) > 0$, $|-K| \ni D \geqslant 0$, $|nK| \ni E \geqslant 0$, $nD + E \equiv 0 \;\Rightarrow$ $D \equiv 0$, contradiction. Hence $h^0(2K) > 0$.

Riemann-Roch: $h^0(3K) + h^0(-2K) \geqslant 1$. Suppose $h^0(3K) \geqslant 1$.

$D \in |2K|$, $E \in |3K|$; $3D, 2E \in |6K| \;\Rightarrow\; 3D = 2E \Rightarrow$ $D = 2F, E = 3F$ with $F \geqslant 0$. But $F \equiv E - D \equiv K$, contradiction.

Therefore $h^0(-2K) > 0$, and $2K \equiv 0$. ∎

The double cover of an Enriques surface

Let S be an Enriques surface. View $\mathcal{K}_S$ as a line bundle $p : \mathcal{K} \to S$; we have a non-vanishing section ω of $H^0(2K)$. Let

$$X = \{x \in \mathcal{K} \,|\, x^2 = \omega(px)\}$$

It is a closed subvariety of $\mathcal{K}$; for each $y \in S$ there are 2 points in X above y, exchanged by the involution $\sigma : x \mapsto -x$. This involution acts freely, and p_X identifies S with X/σ.

The morphism $p_X : X \to S$ is étale, hence $p_X^* \mathcal{K}_S \cong \mathcal{K}_X$.

Consider the pull back diagram:

$$\begin{array}{ccc} \mathcal{K} \times_S \mathcal{K} & \longrightarrow & \mathcal{K} \\ {\scriptstyle p'} \downarrow & & \downarrow {\scriptstyle p} \\ \mathcal{K} & \xrightarrow{\ p\ } & S \end{array}$$

p' has a canonical section $x \mapsto (x, x)$; this section does not vanish outside the zero section of $\mathcal{K}$. Therefore $p^* \mathcal{K}_{|S} = \mathcal{K}_X$ is trivial.

We will admit $q = 0$, so X is a K3 surface.

Examples

- $S_4 \subset \mathbb{P}^3$ (smooth) is a K3 surface.

Indeed $K_{S_d} \equiv (d-4)H$, so $\equiv 0$ for $d = 4$. To prove $q = 0$ we admit a classical result:

Lemma

$H^i(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k)) = 0$ for all k and $0 < i < n$.

Then from the exact sequence $0 \to \mathcal{O}_{\mathbb{P}^3}(-4) \to \mathcal{O}_{\mathbb{P}^3} \to \mathcal{O}_S \to 0$ we get $H^1(\mathcal{O}_S) = 0$. ∎

- More generally, for each $g \geqslant 3$, there is a family of K3 surfaces of degree $2g-2$ in $\mathbb{P}^g$: in $\mathbb{P}^4$ we get the intersection of a quadric and a cubic, in $\mathbb{P}^5$ the intersection of 3 quadrics, etc. These surfaces have a rich geometry and have been, and still are, extensively studied.

An Enriques surface

In $\mathbb{P}^5$, with homogeneous coordinates $X_0, X_1, X_2, X_0', X_1', X_2'$, consider the surface S defined by

$$P(X) + P'(X') = Q(X) + Q'(X') = R(X) + R'(X') = 0\,,$$

where $P, Q, R; P', Q', R'$ are general quadratic forms in 3 variables. The involution $\sigma : (X_i, X_j') \mapsto (-X_i, X_j')$ preserves S; its fixed points are the 2-planes $X_i = 0$ and $X_j' = 0$, which are not on S since the quadratic forms are general. The surface quotient S/σ is an Enriques surface.