COMMUTATIVE ALGEBRA NOTES

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Contents

1. Introduction

In this lecture, we consider a (Noetherian) commutative ring R with identity element.

I will assume that students know about basic definitions and properties of rings, ideals, modules, morphisms (e.g. Chapter 1–3 of [1]). Our main textbook is [2]. We will cover selected topics in order to serve the lecture of geometry of syzygies ([2, Section 17–19]).

1.1. Nakayama's lemma.

The *Jacobson radical* $J(R)$ of R is the intersection of all maximal ideals. Note that $y \in J(R)$ iff $1 - xy$ is a unit in R for every $x \in R$.

Theorem 1.1 (Nakayama's lemma)**.** *Let I be an ideal contained in the Jacobson radical of R, and M a finitely generated R-module. If $IM = M$, then $M = 0$.*

Lemma 1.2. *Let I be an R-ideal and M a finitely generated R-module. If $IM = M$, then there exists $y \in I$ such that $(1 - y)M = 0$.*

Proof. This is a consequence of the Caylay–Hamilton theorem. Consider $m_1, \dots, m_n$ a set of generators in M, then there exists an $n \times n$ matrix A with coefficients in I such that $(m_1, \dots, m_n)^T = A(m_1, \dots, m_n)^T$. Set $\mathbf{m} = (m_1, \dots, m_n)^T$. Hence $(I_n - A)\mathbf{m} = 0$. Note that $\mathrm{adj}(I_n - A)(I_n - A) =$

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$\det(I_n - A)I_n$, we know that $\det(I_n - A)\mathbf{m} = 0$, that is, $\det(I_n - A)m_i = 0$ for all i. This implies that $\det(I_n - A)M = 0$. □

Example 1.3. If we do not assume that M is finitely generated, this is not true. For example, consider $R = k[[x]]$, $M = k[[x, x^{-1}]]$.

Corollary 1.4. *Let I be an ideal contained in the Jacobson radical of R, and M a finitely generated R-module. If $N + IM = M$ for some submodule $N \subset M$, then $M = N$.*

Proof. Apply Nakayama's lemma to M/N. □

Corollary 1.5. *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module. Consider $m_1, \dots, m_n \in M$. If $\bar{m}_1, \dots, \bar{m}_n \in M/\mathfrak{m}M$ is a basis (as a $R/\mathfrak{m}$-vector space), then $m_1, \dots, m_n$ generates M (which is also a minimal set of generators.)*

Proof. Apply Corollary 1.4 to N the submodule generated by $m_1, \dots, m_n$. □

1.2. **Noetherian rings.**

Definition 1.6 (Noetherian ring)**.** A ring R is *Noetherian* if one of the following equivalent conditions holds:

(1) Every non-empty set of ideals has a maximal element;
(2) The set of ideals satisfies the ascending chain condition (ACC);
(3) Every ideal is finitely generated.

In this lecture, we assume all rings are Noetherian and all modules are finitely generated for simplicity.

Theorem 1.7 (Hilbert basis theorem)**.** *If R is Noetherian, then $R[x]$ is Noetherian.*

Idea of proof. Consider $I \subset R[x]$ an ideal. Consider $J \subset R$ the leading coefficients of I, then J is finitely generated. We may assume that J is generated by the leading coefficients of $f_1, \dots, f_n \in R[x]$. Take I' be the ideal generated by $f_1, \dots, f_n$, then it is easy to see that any $f \in I$ can be written as $f = f' + g$ with $f' \in I'$ and $\deg g < \max_i\{\deg f_i\} = r$. So

$$I = I \cap (R \oplus Rx \oplus \cdots \oplus Rx^{r-1}) + I'$$

is finitely generated. (Check that $I \cap (R \oplus Rx \oplus \cdots \oplus Rx^{r-1})$ is finitely generated!) □

Example 1.8. Any quotient of polynomial ring $k[x_1, \dots, x_n]/I$ is Noetherian.

1.3. **Associated primes.**

We will use the notion $(A : B)$ to define the set $\{a \mid aB \subset A\}$ whenever it makes sense. For example, if $N, N' \subset M$ are R-modules and I an ideal, then we can define $(N : I)$ as a submodule of M, and $(N' : N)$ an ideal. Usually the set $(0 : N)$ is denoted by $\mathrm{ann}(N)$ and called the *annihilator* of N, that is, the set of elements whose multiplication action kills N.

Definition 1.9 (Associated prime)**.** A prime P of R is *associated* to M if $P = \mathrm{ann}(x)$ for some $x \in M$.

Associated primes are important in the primary decomposition. But here we mainly focus on its relation with zero-divisors.

Theorem 1.10. *Let R be a Noetherian ring and M a finitely generated R-module. Then the union of associated primes to M consists of zero and zero-divisors. Moreover, there are only finitely many associated primes.*

Proof. We want to show that

$$\bigcup_{\mathrm{ann}(x):\mathrm{prime}} \mathrm{ann}(x) = \bigcup_{x\neq 0} \mathrm{ann}(x).$$

So it suffices to show that if $\mathrm{ann}(y)$ is maximal among all $\mathrm{ann}(x)$, then $\mathrm{ann}(y)$ is prime. Consider $rs \in \mathrm{ann}(y)$ such that $s \notin \mathrm{ann}(y)$, then $rsy = 0$ but $sy \neq 0$. We know that $\mathrm{ann}(y) \subset \mathrm{ann}(sy)$, so equality holds by maximality. This implies that $r \in \mathrm{ann}(y)$.

To prove the finiteness, we only outline the idea here. Denote $\mathrm{Ass}(M)$ the set of associated primes. Then it is not hard to see that for a short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

we have

$$\mathrm{Ass}(M') \subset \mathrm{Ass}(M) \subset \mathrm{Ass}(M') \cup \mathrm{Ass}(M'').$$

So inductively we get the finiteness. □

Remark 1.11. Another fact is that if P is a prime minimal among all primes containing $\mathrm{ann}(M)$, then P is an associated prime.

Corollary 1.12. *Let R be a Noetherian ring and M a finitely generated R-module. Let I be an ideal. Then either I contains a non zero-divisor on M, or I annihilated a non-zero element of M.*

Proof. Suppose that I contains only zero-divisors on M, then by Theorem 1.10, $I \subset \bigcup_{\mathrm{ann}(x):\mathrm{prime}} \mathrm{ann}(x)$. So the conclusion follows from the following easy lemma. □

Lemma 1.13. *Let I be an ideal and let $P_1, \dots, P_n$ be primes of R. If $I \subset \bigcup_i P_i$, then $I \subset P_i$ for some i.*

1.4. **Tensor products and Tor.** Let M, N be R-modules, the *tensor product* $M \otimes N$ is defined by the module generated by

$$\{m \otimes n \mid m \in M, n \in N\},$$

modulo relations

$$(m + m') \otimes n = m \otimes n + m' \otimes n;$$
$$m \otimes (n + n') = m \otimes n + m \otimes n';$$
$$(rm) \otimes n = m \otimes (rn) = r(m \otimes n)$$

for $m \in M, n \in N, r \in R$. It can be characterized by the universal property that if $f : M \times N \to P$ is an R-bilinear map, then there exists a unique $g : M \otimes N \to P$ such that f factors through g.

Example 1.14. (1) $M \otimes R \simeq M$, $M \otimes R^n \simeq M^n$;
(2) $M \otimes R/I \simeq M/IM$;

(3) $(M \otimes_R N)_P \simeq M_P \otimes_{R_P} N_P$.

Proposition 1.15. $(- \otimes N)$ *is a right-exact functor. If*

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is a exact sequence of R-modules, then

$$M' \otimes N \xrightarrow{f \otimes 1} M \otimes N \xrightarrow{g \otimes 1} M'' \otimes N \to 0$$

is exact.

Definition 1.16 (Flat module)**.** N is *flat* if $(- \otimes N)$ is an exact functor, that is, if

$$0 \to M' \to M \to M'' \to 0$$

is a exact sequence of R-modules, then

$$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$$

is exact.

To study flatness, we need to introduce Tor from homological algebra.

Definition 1.17 (Projective module)**.** An R-module M is *projective* if for any surjective map $f : N_1 \to N_2$ and any map $g : M \to N_2$, there exists $h : M \to N_1$ such that $f \circ h = g$.

Example 1.18. Free modules are flat and projective.

Definition 1.19 (Complexes and homologies)**.** A *complex* of R-modules is a sequence of R-modules with (differential) homomorphisms

$$\mathcal{F} : \cdots \to F_{i+1} \xrightarrow{\delta_{i+1}} F_i \xrightarrow{\delta_i} F_{i-1} \to \dots$$

such that $\delta_i \delta_{i+1} = 0$ for each i. Denote the *homology* to be $H_i(\mathcal{F}) = \ker(\delta_i)/\mathrm{im}(\delta_{i+1})$. We say that $\mathcal{F}$ is *exact* at degree i if $H_i(\mathcal{F}) = 0$. A morphism of complexes $\phi : \mathcal{F} \to \mathcal{G}$ is given by $\phi_i : F_i \to G_i$ commuting with differentials, that is, we have a commutative diagram

$$\begin{array}{cccccccccc}
\mathcal{F}: & \dots & \longrightarrow & F_{i+1} & \longrightarrow & F_i & \longrightarrow & F_{i-1} & \longrightarrow & \dots \\
& & & \downarrow{\scriptstyle \phi_{i+1}} & & \downarrow{\scriptstyle \phi_i} & & \downarrow{\scriptstyle \phi_{i-1}} & & \\
\mathcal{G}: & \dots & \longrightarrow & G_{i+1} & \longrightarrow & G_i & \longrightarrow & G_{i-1} & \longrightarrow & \dots
\end{array}$$

This naturally gives morphisms between homologies $\phi_i : H_i(\mathcal{F}) \to H_i(\mathcal{G})$.

Definition 1.20 (Projective resolution)**.** A *projective resolution* of an R-module M is a complex of projective modules

$$\mathcal{F} : \cdots \to F_n \to \cdots \to F_1 \xrightarrow{\phi_1} F_0$$

which is exact and $\mathrm{coker}(\phi_1) = M$. Sometimes we also denote it by

$$\mathcal{F} : \cdots \to F_n \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 (\to M \to 0).$$

Definition 1.21 (Left derived functor)**.** Let T be a right-exact functor. Given a projective resolution of an R-module M:

$$\mathcal{F} : \cdots \to F_n \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 (\to M \to 0).$$

Define the *left derived functor* by $L_i T(M) := H_i(T\mathcal{F})$, which is just the homology of

$$T\mathcal{F} : \cdots \to T(F_n) \to \cdots \to T(F_1) \to T(F_0)(\to T(M) \to 0).$$

We collect basic properties of derived functors here.

Proposition 1.22. (1) *$L_0 T(M) = T(M)$;*

(2) *$L_i T(M)$ is independent of the choice of projective resolution;*
(3) *If M is projective, then $L_i T(M) = 0$ for $i > 0$.*
(4) *For a short exact sequence of R-modules*

$$0 \to A \to B \to C \to 0,$$

we have a long exact sequence

$$\begin{aligned} &\dots \\ &\to L_3T(A) \to L_3T(B) \to L_3T(C) \\ &\to L_2T(A) \to L_2T(B) \to L_2T(C) \\ &\to L_1T(A) \to L_1T(B) \to L_1T(C) \\ &\to T(A) \to T(B) \to T(C) \to 0. \end{aligned}$$

Definition 1.23 (Tor)**.** For an R-module N, $\mathrm{Tor}_i^R(-, N)$ is defined by $L_i T(-)$ where $T = (- \otimes N)$.

Remark 1.24. So to compute $\mathrm{Tor}_i^R(M, N)$, we should pick a projective resolution $\mathcal{F}$ of M and compute $H_i(\mathcal{F} \otimes N)$. Note that tensor products are symmetric, that is, $M \otimes N \simeq N \otimes M$, it can be seen that $\mathrm{Tor}_i^R(M, N) \simeq \mathrm{Tor}_i^R(N, M)$, and $\mathrm{Tor}_i^R(M, N)$ can be also computed by pick a projective resolution $\mathcal{G}$ of N and compute $H_i(M \otimes \mathcal{G})$.

Theorem 1.25. *TFAE:*

(1) *N is flat;*
(2) *$\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$ and all M;*
(3) *$\mathrm{Tor}_1^R(M, N) = 0$ for all M.*

Proof. (1) $\implies$ (2): take a projective resolution $\mathcal{F}$ of M, we need to compute $H_i(\mathcal{F} \otimes N)$. As N is flat, $\mathcal{F} \otimes N$ is exact, hence $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$.

(2) $\implies$ (3): trivial.

(3) $\implies$ (1): this follows from the long exact sequence

$$\mathrm{Tor}_1^R(M'', N) \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0.$$

□

2. Koszul complexes and regular sequences

2.1. **Regular sequences.**

Definition 2.1 (Regular sequence)**.** Let R be a ring and M an R-module. A sequence of elements $x_1, \dots, x_n \in R$ is called a *regular sequence* on M (or *M-sequence*) if

(1) $(x_1, \dots, x_n)M \neq M$;
(2) For each $1 \leq i \leq n$, x_i is not a zero-divisor on $M/(x_1, \dots, x_{i-1})M$.

Definition 2.2 (Depth)**.** Let R be a ring, I an ideal, and M an R-module. Suppose $IM \neq M$. The *depth* of I on M, $\operatorname{depth}(I, M)$, is defined by the maximal length of M-sequences in I.

Remark 2.3. (1) If $M = R$, then simply denote $\operatorname{depth} I := \operatorname{depth}(I, M)$.
(2) We will see soon (Theorem 2.15) that any maximal M-sequence has the same length.

Example 2.4. If $R = k[x_1, \dots, x_n]$, then $x_1, \dots, x_n$ is a regular sequence. We will see soon that $\operatorname{depth}(x_1, \dots, x_n) = n$.

Remark 2.5. The depth measures the size of an ideal, and an element in the regular sequence corresponds to a hypersurface in geometry. So a regular sequence in I corresponds to a set of hypersurface containing $V(I)$ intersecting each other "properly". Consider for example $R = k[x, y]$ or $k[x, y]/(xy)$, $I = (x, y)$.

2.2. **Koszul complexes.**

Definition 2.6 (Complexes and homologies)**.** A *complex* of R-modules is a sequence of R-modules with homomorphisms

$$\mathcal{F} : \dots \to M_{i-1} \xrightarrow{\delta_{i-1}} M_i \xrightarrow{\delta_i} M_{i+1} \to \dots$$

such that $\delta_i \delta_{i-1} = 0$ for each i. Denote the *(co)homology* to be $H^i(\mathcal{F}) = \ker(\delta_i)/\operatorname{im}(\delta_{i-1})$.

We will introduce Koszul complexes and explain how regular sequences are related to Koszul complexes.

Example 2.7 (Koszul complex of length 1)**.** Given $x \in R$. The Koszul complex of length 1 is given by

$$K(x) : 0 \to R \xrightarrow{x} R \to 0.$$

Note that $H^0(K(x)) = (0 : x)$, $H^1(K(x)) = R/xR$. Then x is an R-sequence if (1) $H^1(K(x)) \neq 0$; (2) $H^0(K(x)) = 0$.

Example 2.8 (Koszul complex of length 2)**.** Given $x, y \in R$. The Koszul complex of length 2 is given by

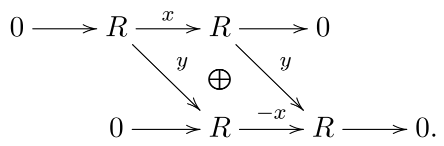
$$K(x, y) : 0 \to R \xrightarrow{\begin{pmatrix} y \\ x \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} -x & y \end{pmatrix}} R \to 0.$$

Note that $H^0(K(x, y)) = (0 : (x, y))$. $H^2(K(x, y)) = R/(x, y)R$. We can compute $H^1(K(x, y))$ (Exercise). It turns out that if x is not a zero-divisor in R, then $H^1(K(x, y)) \simeq (x : y)/(x)$. So $H^1(K(x, y)) = 0$ if and only if

y is not a zero-divisor of $R/(x)$. In conclusion, x, y is an R-sequence if (1) $H^2(K(x,y)) \neq 0$; (2) $H^0(K(x,y)) = H^1(K(x,y)) = 0$.

Theorem 2.9. *Let $(R, \mathfrak{m})$ be a local ring and $x, y \in \mathfrak{m}$. Then x, y is a regular sequence iff $H^1(K(x,y)) = 0$. In particular, x, y is a regular sequence iff y, x is a regular sequence.*

Proof. This is not a direct consequence of the above argument, as we need to show that x is a non-zero-divisor (equivalent to $H^0(K(x)) = 0$). Write $K(x,y)$ as the following:



Then this gives a short exact sequence of complexes

$$\begin{array}{rcccccccc}
K(x)[-1]: & & 0 & \longrightarrow & R & \xrightarrow{-x} & R & \longrightarrow & 0\ . \\
& & \downarrow & & \downarrow{\scriptstyle i_2} & & \downarrow{\scriptstyle 1} & & \\
K(x,y): 0 & \longrightarrow & R & \longrightarrow & R^2 & \longrightarrow & R & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle 1} & & \downarrow{\scriptstyle p_1} & & \downarrow & & \\
K(x): 0 & \longrightarrow & R & \xrightarrow{x} & R & \longrightarrow & 0 & &
\end{array}$$

That is,

$$0 \to K(x)[-1] \to K(x,y) \to K(x) \to 0.$$

Then this induces a long exact sequences of homologies

$$H^0(K(x)) \xrightarrow{y} H^0(K(x)) \to H^1(K(x,y)) \to H^1(K(x)).$$

So $H^1(K(x,y)) = 0$ implies that $yH^0(K(x)) = H^0(K(x))$, which means that $H^0(K(x)) = 0$ by Nakayama's lemma. □

Corollary 2.10. *Let $(R, \mathfrak{m})$ be a local ring and $x_1, \dots, x_n \in \mathfrak{m}$. Suppose that $x_1, \dots, x_n$ is a regular sequence, then any permutation of $x_1, \dots, x_n$ is again a regular sequence. (Exercise.)*

We will define Koszul complexes and show this correspondence in general.

Definition 2.11 (Exterior algebra)**.** Let N be an R-module. Denote the *tensor algebra*

$$T(N) = R \oplus N \oplus (N \otimes N) \oplus \dots$$

The *exterior algebra* $\bigwedge N = \oplus_m \bigwedge^m N$ is defined by $T(N)$ modulo the relations $x \otimes x$ (and hence $x \otimes y + y \otimes x$) for $x, y \in N$. The product of $a, b \in \bigwedge N$ is written as $a \wedge b$.

Definition 2.12 (Koszul complex)**.** Let N be an R-module, $x \in N$. Define the *Koszul complex* to be

$$K(x): 0 \to R \to N \to \bigwedge^2 N \to \dots \to \bigwedge^i N \xrightarrow{d_x} \bigwedge^{i+1} N \to \dots$$

where d_x sends a to $x \wedge a$. If $N \simeq R^n$ is a free module of rank n (we always consider this situation) and $x = (x_1, \dots, x_n) \in R^n$, then we denote $K(x)$ by $K(x_1, \dots, x_n)$.

Remark 2.13. (1) The $R \to N$ maps 1 to x.

(2) Consider $N = R^2$ (with basis e_1, e_2) and $x = (x_1, x_2)$, then $\bigwedge^2 N \simeq R$ (with bases $e_1 \wedge e_2$), and the map $N \to \bigwedge^2 N$ is given by $e_1 \mapsto (x_1e_1 + x_2e_2) \wedge e_1 = -x_2e_1 \wedge e_2$ and $e_2 \mapsto x_1e_1 \wedge e_2$. In other words,

$$K(x_1, x_2) : 0 \to R \xrightarrow{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}} R^{\oplus 2} \xrightarrow{\begin{pmatrix} -x_2 & x_1 \end{pmatrix}} R \to 0.$$

Example 2.14. $H^n(K(x_1, \dots, x_n)) = R/(x_1, \dots, x_n)$. Consider the corresponding complex

$$\bigwedge^{n-1} N \xrightarrow{d_x} \bigwedge^n N \to \bigwedge^{n+1} N = 0$$

Denote $e_1, \dots, e_n$ to be a basis of $N \simeq R^n$, then the basis of $\bigwedge^n N$ is just $e_1 \wedge \cdots \wedge e_n$, and the basis of $\bigwedge^{n-1} N$ is $e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n$ $(1 \leq i \leq n)$. d_x maps $e_1 \wedge \cdots \wedge \hat{e}_i \wedge \cdots \wedge e_n$ to $(-1)^{i-1}x_ie_1 \wedge \cdots \wedge e_n$. So $\mathrm{im} d_x = (x_1, \dots, x_n)$ and $H^n(K(x_1, \dots, x_n)) = R/(x_1, \dots, x_n)$.

2.3. **Koszul complexes versus regular sequences.** Now we can state the main theorem of this section.

Theorem 2.15. *Let M be a finitely generated R-module. If*

$$H^j(M \otimes K(x_1, \dots, x_n)) = 0$$

for $j < r$ and $H^r(M \otimes K(x_1, \dots, x_n)) \neq 0$, then every maximal M-sequence in $I = (x_1, \dots, x_n) \subset R$ has length r.

Idea of proof. Firstly, we consider the case that $x_1, \dots, x_s$ is a maximal M-sequence. In this case it is natural to prove this case by induction on n and s.

In order to reduce the general case to this case, we consider $y_1, \dots, y_s$ a maximal M-sequence, and consider $H^j(M \otimes K(y_1, \dots, y_s, x_1, \dots, x_n))$.

So to deal with both cases, we need to investigate the relation between $K(y_1, \dots, y_s, x_1, \dots, x_n)$ and $K(x_1, \dots, x_n)$ and the relation of their homologies. □

Corollary 2.16. *If $x_1, \dots, x_n$ is an M-sequence, then $H^j(M \otimes K(x_1, \dots, x_n)) = 0$ for $j < n$ and $H^n(M \otimes K(x_1, \dots, x_n)) = M/(x_1, \dots, x_n)M$.*

Proof. By definition, $\mathrm{depth}(I, M) \geq n$, so $H^j(M \otimes K(x_1, \dots, x_n)) = 0$ for $j < n$. On the other hand,

$$\begin{aligned} H^n(M \otimes K(x_1, \dots, x_n)) &= \mathrm{coker}(M \otimes \bigwedge^{n-1} N \to M \otimes \bigwedge^n N) \\ &= M \otimes \mathrm{coker}(\bigwedge^{n-1} N \to \bigwedge^n N) \\ &= M \otimes R/(x_1, \dots, x_n) = M/(x_1, \dots, x_n)M. \end{aligned}$$

Here we use the fact that $M \otimes -$ is right-exact. □

Theorem 2.15 can be strengthen for local rings.

Theorem 2.17. *Let $(R, \mathfrak{m})$ be a local ring, $x_1, \dots, x_n \in \mathfrak{m}$. Let M be a finitely generated R-module. If $H^k(M \otimes K(x_1, \dots, x_n)) = 0$ for some k, then $H^j(M \otimes K(x_1, \dots, x_n)) = 0$ for all $j < r$. Moreover, if $H^{n-1}(M \otimes K(x_1, \dots, x_n)) = 0$, then $x_1, \dots, x_n$ is an M-sequence.*

Corollary 2.18. *If R is local and $(x_1, \dots, x_n)$ is a proper ideal containing an M-sequence of length n, then $x_1, \dots, x_n$ is an M-sequence.*

Proof. $H^n(M \otimes K(x_1, \dots, x_n)) = M/(x_1, \dots, x_n)M \neq 0$ by Nakayama's lemma. Take r minimal such that $H^r(M \otimes K(x_1, \dots, x_n)) \neq 0$, then every maximal M-sequence in $(x_1, \dots, x_n)$ has length r, which implies that $r \geq n$. So $H^{n-1}(M \otimes K(x_1, \dots, x_n)) = 0$ and $x_1, \dots, x_n$ is an M-sequence. □

2.4. **Operations on Koszul complexes.**

Definition 2.19 (Tensor product of two complexes)**.** Given two complexes

$$\mathcal{F} : \dots \to F_i \xrightarrow{\phi_i} F_{i+1} \to \dots;$$

$$\mathcal{G} : \dots \to G_i \xrightarrow{\psi_i} G_{i+1} \to \dots$$

define the tensor product

$$\mathcal{F} \otimes \mathcal{G} : \dots \to \bigoplus_{i+j=k} F_i \otimes G_j \xrightarrow{d_k} \bigoplus_{i+j=k+1} F_i \otimes G_j \to \dots,$$

where the map $F_i \otimes G_j \to F_{i'} \otimes G_{j'}$ is $\begin{cases} \phi_i \otimes 1 & \text{if } i' = i+1; \\ (-1)^i 1 \otimes \psi_j & \text{if } j' = j+1; \\ 0 & \text{otherwise.} \end{cases}$ (Check $dd = 0$.)

Definition 2.20 (Shift)**.** Given a complex

$$\mathcal{F} : \dots \to F_i \xrightarrow{\phi_i} F_{i+1} \to \dots;$$

Denote $\mathcal{F}[n]$ to be the complex obtained by shifting $\mathcal{F}$ (to the left) n times. That is, $\mathcal{F}[n]_i = \mathcal{F}_{n+i}$, and the differential is multiplied by $(-1)^n$. Denote $R[n]$ to be the simple complex whose n-th position is R. Note that $\mathcal{F}[n] = R[n] \otimes \mathcal{F}$.

Definition 2.21 (Mapping cone)**.** For $y \in R$, consider $\mathcal{F} = K(y)$, that is,

$$\mathcal{F} : 0 \to R \xrightarrow{y} R \to 0.$$

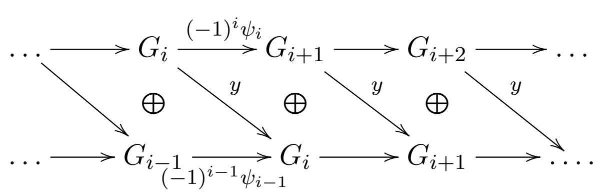
Then there is a natural exact sequence of complexes

$$0 \to R[-1] \to \mathcal{F} \to R \to 0.$$

Tensoring a complex $\mathcal{G}$, this gives an exact sequence

$$0 \to \mathcal{G}[-1] \to \mathcal{F} \otimes \mathcal{G} \to \mathcal{G} \to 0.$$

Here $\mathcal{F} \otimes \mathcal{G}$ is the *mapping cone* of the map $\mathcal{G} \xrightarrow{y} \mathcal{G}$, in fact, it is given by



From this exact sequence, we get a long exact sequence of homologies

$$\dots \to H^{i-1}(\mathcal{G}) \xrightarrow{y} H^{i-1}(\mathcal{G}) \to H^i(\mathcal{F} \otimes \mathcal{G}) \to H^i(\mathcal{G}) \xrightarrow{y} \dots.$$

Here note that $H^{i-1}(\mathcal{G}) = H^i(\mathcal{G}[-1])$.

Proposition 2.22. *If $N = N' \oplus N''$, then $\bigwedge N = \bigwedge N' \otimes \bigwedge N''$. If $x' \in N$ and $x'' \in N''$, take $x = (x', x'') \in N$, then $K(x) = K(x') \otimes K(x'')$.*

Proof. Note that here the (skew-commutative) algebra structure of $\bigwedge N' \otimes \bigwedge N''$ is given by

$$(a \otimes b) \wedge (a' \otimes b') = (-1)^{\deg a' \deg b}((a \wedge a') \otimes (b \wedge b'))$$

for homogenous elements. This is just linear algebra. It suffices to check the differentials coincide, that is, for $y' \in \bigwedge N'$, $y'' \in \bigwedge N''$, $x \wedge (y' \otimes y'') = (x' \otimes 1 + 1 \otimes x'') \wedge (y' \otimes y'') = (x' \wedge y') \otimes y'' + (-1)^{\deg y'} y' \otimes (x'' \wedge y'')$. □

Corollary 2.23. *If $y_1, \ldots, y_r$ are elements in $(x_1, \ldots, x_n)$ and M is an R-module, then*

$$H^*(M \otimes K(x_1, \ldots, x_n, y_1, \ldots, y_r)) \simeq H^*(M \otimes K(x_1, \ldots, x_n)) \otimes \bigwedge R^r$$

as graded modules, which means that

$$H^i(M \otimes K(x_1, \ldots, x_n, y_1, \ldots, y_r)) \simeq \bigoplus_{j+k=i} H^j(M \otimes K(x_1, \ldots, x_n)) \otimes \bigwedge^k R^r.$$

So $H^i(M \otimes K(x_1, \ldots, x_n, y_1, \ldots, y_r)) = 0$ iff $H^j(M \otimes K(x_1, \ldots, x_n)) = 0$ for any $i - r \leq j \leq i$.

Proof. As $y_1, \ldots, y_r$ are elements in $(x_1, \ldots, x_n)$, there is an isomorphism

$$R^n \oplus R^r \simeq R^n \oplus R^r$$

sending $(x_1, \ldots, x_n, y_1, \ldots, y_n)$ to $(x_1, \ldots, x_n, 0, \ldots, 0)$. So by functoriality of Koszul complex,

$$\begin{aligned} K(x_1, \ldots, x_n, y_1, \ldots, y_r) &\simeq K(x_1, \ldots, x_n, 0, \ldots, 0) \\ &\simeq K(x_1, \ldots, x_n) \otimes K(0, \ldots, 0). \end{aligned}$$

Here

$$K(0, \ldots, 0) : 0 \to R \xrightarrow{0} \bigwedge^2 R^r \xrightarrow{0} \ldots \xrightarrow{0} \bigwedge^r R^r \to 0.$$

□

Corollary 2.24. *If $x = (x', y) \in N = N' \oplus R$, then $K(x)$ is isomorphic to the mapping cone of $K(x') \xrightarrow{y} K(x')$. In particular, we have a long exact sequence*

$$\begin{aligned} \cdots \to H^i(M \otimes K(x')) \xrightarrow{y} H^i(M \otimes K(x')) \to H^{i+1}(M \otimes K(x)) \to \\ \to H^{i+1}(M \otimes K(x')) \xrightarrow{y} H^{i+1}(M \otimes K(x')) \to \ldots \end{aligned}$$

Proof. Note that $N' \oplus R \simeq R \oplus N'$. Hence $K(x) \simeq K(y, x') = K(y) \otimes K(x')$. This gives a short exact sequence

$$0 \to K(x')[-1] \to K(x) \to K(x') \to 0.$$

Tensoring with M, we get

$$0 \to M \otimes K(x')[-1] \to M \otimes K(x) \to M \otimes K(x') \to 0.$$

(Why exact?). □

2.5. **Proof of the main theorems.** The following is a more precise version.

Corollary 2.25. *If $x_1, \dots, x_i$ is an M-sequence, then*

$$H^i(M \otimes K(x_1, \dots, x_n)) = ((x_1, \dots, x_i)M : (x_1, \dots, x_n))/(x_1, \dots, x_i)M.$$

In particular, in this case, $H^j(M \otimes K(x_1, \dots, x_n)) = 0$ for $j < i$. If $IM \neq M$ ($I = (x_1, \dots, x_n)$) and $x_1, \dots, x_i$ is a maximal M-sequence, then $H^i(M \otimes K(x_1, \dots, x_n)) \neq 0$.

Proof. We do induction on i. If $i = 0$ this is trivial. If $i > 0$, then we do induction on n. If $n = i$, this follows easily by Example 2.14. If $n > i$, then by Corollary 2.24, there is an exact sequence

$$H^{i-1}(M \otimes K(x_1, \dots, x_{n-1})) \to H^i(M \otimes K(x_1, \dots, x_n)) \to$$
$$\to H^i(M \otimes K(x_1, \dots, x_{n-1})) \xrightarrow{x_n} H^i(M \otimes K(x_1, \dots, x_{n-1}))$$

Here by induction,

$$H^{i-1}(M \otimes K(x_1, \dots, x_{n-1})) = ((x_1, \dots, x_{i-1})M : (x_1, \dots, x_{n-1}))/(x_1, \dots, x_{i-1})M = 0$$

as x_i is not a zeo-divisor of $M/(x_1, \dots, x_{i-1})M$ (this also proves the second statement). Hence $H^i(M \otimes K(x_1, \dots, x_n))$ is just the kernel of

$$H^i(M \otimes K(x_1, \dots, x_{n-1})) \xrightarrow{x_n} H^i(M \otimes K(x_1, \dots, x_{n-1})).$$

By induction,

$$H^i(M \otimes K(x_1, \dots, x_{n-1})) = ((x_1, \dots, x_i)M : (x_1, \dots, x_{n-1}))/(x_1, \dots, x_i)M,$$

so it easy to compute the kernel.

To show the last statement, note that I is contained in the set of zero-divisors on $M/(x_1, \dots, x_i)M$, so I is contained in the union of associated primes and hence $I \subset \operatorname{ann}(x)$ for some non-zero $x \in M/(x_1, \dots, x_i)M$ by Corollary 1.12. This implies that $((x_1, \dots, x_i)M : I)/(x_1, \dots, x_i)M \neq 0$. □

Proof of Theorem 2.15. Let $y_1, \dots, y_s$ be a maximal M-sequence and r be the minimal such that

$$H^r(M \otimes K(x_1, \dots, x_n)) \neq 0.$$

The goal is to show that $r = s$.

By Corollary 2.23, r is the minimal such that

$$H^r(M \otimes K(x_1, \dots, x_n, y_1, \dots, y_s)) \neq 0.$$

If $IM \neq M$, then by Corollary 2.25, $r = s$. So it suffices to show that $IM \neq M$. This follows from Lemma 2.26(2) and the nonvanishing of homologies. □

Lemma 2.26. (1) *If $y \in (x_1, \dots, x_n)$, then $H^j(M \otimes K(x_1, \dots, x_n))$ is annihilated by y for any M and any j.*
(2) *If $(x_1, \dots, x_n)M = M$, then $H^j(M \otimes K(x_1, \dots, x_n)) = 0$ for any j.*

Proof. (1) Here we give a different proof from the book (which uses dual Koszul complex). Note that by Corollary 2.24, there is a long exact sequence

$$H^j(M \otimes K(x_1, \dots, x_n, y)) \to H^j(M \otimes K(x_1, \dots, x_n)) \xrightarrow{y} H^j(M \otimes K(x_1, \dots, x_n)).$$

So the statement is equivalent to that the first arrow is surjective. By the proof of Corollary 2.23, this arrow splits.

(2) Replacing R by $R/\mathrm{ann}(M)$ will not change $M \otimes K(x_1, \ldots, x_n)$, so we may assume that $\mathrm{ann}(M) = 0$. By $(x_1, \ldots, x_n)M = M$ and Lemma 1.2, there is $y \in (x_1, \ldots, x_n)$ such that $(1 - y)M = 0$, which implies that $y = 1 \in (x_1, \ldots, x_n)$. Then apply (1). □

Proof of Theorem 2.17. We prove the first statement by induction on n. Suppose $H^k(M \otimes K(x_1, \ldots, x_n)) = 0$, then by Corollary 2.24,

$$H^{k-1}(M \otimes K(x_1, \ldots, x_{n-1})) \xrightarrow{x_n} H^{k-1}(M \otimes K(x_1, \ldots, x_{n-1}))$$

is surjective. Then by Nakayama's lemma, $H^{k-1}(M \otimes K(x_1, \ldots, x_{n-1})) = 0$. By induction, $H^j(M \otimes K(x_1, \ldots, x_{n-1})) = 0$ for $j \leq k-1$. By the long exact sequence in Corollary 2.24, $H^j(M \otimes K(x_1, \ldots, x_n)) = 0$ for $j \leq k - 1$.

We prove the second statement by induction on n. Suppose $H^{n-1}(M \otimes K(x_1, \ldots, x_n)) = 0$, then as above, $H^{n-2}(M \otimes K(x_1, \ldots, x_{n-1})) = 0$, which implies that $x_1, \ldots, x_{n-1}$ is an M-sequence by induction. Then by Corollary 2.25,

$$0 = H^{n-1}(M \otimes K(x_1, \ldots, x_n)) = ((x_1, \ldots, x_{n-1})M : (x_1, \ldots, x_n))/(x_1, \ldots, x_{n-1})M,$$

which implies that x_n is not a zero-divisor of $M/(x_1, \ldots, x_{n-1})M$. □

References

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