COMMUTATIVE ALGEBRA NOTES

CHEN JIANG

Contents

1. Introduction

In this lecture, we consider a (Noetherian) commutative ring R with identity element.

I will assume that students know about basic definitions and properties of rings, ideals, modules, morphisms (e.g. Chapter 1–3 of [1]). Our main textbook is [2]. We will cover selected topics in order to serve the lecture of geometry of syzygies ([2, Section 17–19]).

1.1. Nakayama's lemma.

The *Jacobson radical* $J(R)$ of R is the intersection of all maximal ideals. Note that $y \in J(R)$ iff $1 - xy$ is a unit in R for every $x \in R$.

Theorem 1.1 (Nakayama's lemma)**.** *Let I be an ideal contained in the Jacobson radical of R, and M a finitely generated R-module. If $IM = M$, then $M = 0$.*

Lemma 1.2. *Let I be an R-ideal and M a finitely generated R-module. If $IM = M$, then there exists $y \in I$ such that $(1 - y)M = 0$.*

Proof. This is a consequence of the Caylay–Hamilton theorem. Consider $m_1, \dots, m_n$ a set of generators in M, then there exists an $n \times n$ matrix A with coefficients in I such that $(m_1, \dots, m_n)^T = A(m_1, \dots, m_n)^T$. Set $\mathbf{m} = (m_1, \dots, m_n)^T$. Hence $(I_n - A)\mathbf{m} = 0$. Note that $\operatorname{adj}(I_n - A)(I_n - A) = \det(I_n - A)I_n$, we know that $\det(I_n - A)\mathbf{m} = 0$, that is, $\det(I_n - A)m_i = 0$ for all i. This implies that $\det(I_n - A)M = 0$. □

Example 1.3. If we do not assume that M is finitely generated, this is not true. For example, consider $R = k[[x]]$, $M = k[[x, x^{-1}]]$.

Corollary 1.4. *Let I be an ideal contained in the Jacobson radical of R, and M a finitely generated R-module. If $N + IM = M$ for some submodule $N \subset M$, then $M = N$.*

Date: July 20, 2020.

Proof. Apply Nakayama's lemma to M/N. □

Corollary 1.5. *Let* $(R, \mathfrak{m})$ *be a local ring and* M *a finitely generated* R*-module. Consider* $m_1, \dots, m_n \in M$. *If* $\bar{m}_1, \dots, \bar{m}_n \in M/\mathfrak{m}M$ *is a basis (as a* $R/\mathfrak{m}$*-vector space), then* $m_1, \dots, m_n$ *generates* M *(which is also a minimal set of generators.)*

Proof. Apply Corollary 1.4 to N the submodule generated by $m_1, \dots, m_n$. □

1.2. Noetherian rings.

Definition 1.6 (Noetherian ring)**.** A ring R is *Noetherian* if one of the following equivalent conditions holds:

(1) Every non-empty set of ideals has a maximal element;
(2) The set of ideals satisfies the ascending chain condition (ACC);
(3) Every ideal is finitely generated.

In this lecture, we assume all rings are Noetherian and all modules are finitely generated for simplicity.

Theorem 1.7 (Hilbert basis theorem)**.** *If* R *is Noetherian, then* $R[x]$ *is Noetherian.*

Idea of proof. Consider $I \subset R[x]$ an ideal. Consider $J \subset R$ the leading coefficients of I, then J is finitely generated. We may assume that J is generated by the leading coefficients of $f_1, \dots, f_n \in R[x]$. Take I' be the ideal generated by $f_1, \dots, f_n$, then it is easy to see that any $f \in I$ can be written as $f = f' + g$ with $f' \in I'$ and $\deg g < \max_i\{\deg f_i\} = r$. So

$$I = I \cap (R \oplus Rx \oplus \cdots \oplus Rx^{r-1}) + I'$$

is finitely generated. (Check that $I \cap (R \oplus Rx \oplus \cdots \oplus Rx^{r-1})$ is finitely generated!) □

Example 1.8. Any quotient of polynomial ring $k[x_1, \dots, x_n]/I$ is Noetherian.

1.3. **Associated primes.** We will use the notion $(A : B)$ to define the set $\{a \mid aB \subset A\}$ whenever it makes sense. For example, if $N, N' \subset M$ are R-modules and I an ideal, then we can define $(N : I)$ as a submodule of M, and $(N' : N)$ an ideal. Usually the set $(0 : N)$ is denoted by $\operatorname{ann}(N)$ and called the *annihilator* of N, that is, the set of elements whose multiplication action kills N.

Definition 1.9 (Associated prime)**.** A prime P of R is *associated* to M if $P = \operatorname{ann}(x)$ for some $x \in M$.

Associated primes are important in the primary decomposition. But here we mainly focus on its relation with zero-divisors.

Theorem 1.10. *Let* R *be a Noetherian ring and* M *a finitely generated* R*-module. Then the union of associated primes to* M *consists of zero and zero-divisors. Moreover, there are only finitely many associated primes.*

Proof. We want to show that

$$\bigcup_{\mathrm{ann}(x):\mathrm{prime}} \mathrm{ann}(x) = \bigcup_{x\neq 0} \mathrm{ann}(x).$$

So it suffices to show that if $\mathrm{ann}(y)$ is maximal among all $\mathrm{ann}(x)$, then $\mathrm{ann}(y)$ is prime. Consider $rs \in \mathrm{ann}(y)$ such that $s \notin \mathrm{ann}(y)$, then $rsy = 0$ but $sy \neq 0$. We know that $\mathrm{ann}(y) \subset \mathrm{ann}(sy)$, so equality holds by maximality. This implies that $r \in \mathrm{ann}(y)$.

To prove the finiteness, we only outline the idea here. Denote $\mathrm{Ass}(M)$ the set of associated primes. Then it is not hard to see that for a short exact sequence

$$0 \to M' \to M \to M'' \to 0,$$

we have

$$\mathrm{Ass}(M') \subset \mathrm{Ass}(M) \subset \mathrm{Ass}(M') \cup \mathrm{Ass}(M'').$$

So inductively we get the finiteness. □

Remark 1.11. Another fact is that if P is a prime minimal among all primes containing $\mathrm{ann}(M)$, then P is an associated prime.

Corollary 1.12. *Let R be a Noetherian ring and M a finitely generated R-module. Let I be an ideal. Then either I contains a non zero-divisor on M, or I annihilated a non-zero element of M.*

Proof. Suppose that I contains only zero-divisors on M, then by Theorem 1.10, $I \subset \bigcup_{\mathrm{ann}(x):\mathrm{prime}} \mathrm{ann}(x)$. So the conclusion follows from the following easy lemma. □

Lemma 1.13. *Let I be an ideal and let $P_1, \dots, P_n$ be primes of R. If $I \subset \bigcup_i P_i$, then $I \subset P_i$ for some i.*

1.4. **Tensor products and Tor.** Let M, N be R-modules, the *tensor product* $M \otimes N$ is defined by the module generated by

$$\{m \otimes n \mid m \in M, n \in N\},$$

modulo relations

$$(m + m') \otimes n = m \otimes n + m' \otimes n;$$
$$m \otimes (n + n') = m \otimes n + m \otimes n';$$
$$(rm) \otimes n = m \otimes (rn) = r(m \otimes n)$$

for $m \in M, n \in N, r \in R$. It can be characterized by the universal property that if $f : M \times N \to P$ is an R-bilinear map, then there exists a unique $g : M \otimes N \to P$ such that f factors through g.

Example 1.14.
(1) $M \otimes R \simeq M$, $M \otimes R^n \simeq M^n$;
(2) $M \otimes R/I \simeq M/IM$;
(3) $(M \otimes_R N)_P \simeq M_P \otimes_{R_P} N_P$.

Proposition 1.15. *$(- \otimes N)$ is a right-exact functor. If*

$$M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$$

is a exact sequence of R-modules, then

$$M' \otimes N \xrightarrow{f\otimes 1} M \otimes N \xrightarrow{g\otimes 1} M'' \otimes N \to 0$$

is exact.

Definition 1.16 (Flat module)**.** N is *flat* if $(-\otimes N)$ is an exact functor, that is, if

$$0 \to M' \to M \to M'' \to 0$$

is a exact sequence of R-modules, then

$$0 \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0$$

is exact.

To study flatness, we need to introduce Tor from homological algebra.

Definition 1.17 (Projective module)**.** An R-module M is *projective* if for any surjective map $f : N_1 \to N_2$ and any map $g : M \to N_2$, there exists $h : M \to N_1$ such that $f \circ h = g$.

Example 1.18. Free modules are flat and projective.

Definition 1.19 (Complexes and homologies)**.** A *complex* of R-modules is a sequence of R-modules with (differential) homomorphisms

$$\mathcal{F} : \cdots \to F_{i+1} \xrightarrow{\delta_{i+1}} F_i \xrightarrow{\delta_i} F_{i-1} \to \ldots$$

such that $\delta_i \delta_{i+1} = 0$ for each i. Denote the *homology* to be $H_i(\mathcal{F}) = \ker(\delta_i)/\mathrm{im}(\delta_{i+1})$. We say that $\mathcal{F}$ is *exact* at degree i if $H_i(\mathcal{F}) = 0$. A morphism of complexes $\phi : \mathcal{F} \to \mathcal{G}$ is given by $\phi_i : F_i \to G_i$ commuting with differentials, that is, we have a commutative diagram

$$\begin{array}{llllllll}
\mathcal{F}: & \ldots \longrightarrow & F_{i+1} & \longrightarrow & F_i & \longrightarrow & F_{i-1} & \longrightarrow \ldots \\
 & & \downarrow \phi_{i+1} & & \downarrow \phi_i & & \downarrow \phi_{i-1} & \\
\mathcal{G}: & \ldots \longrightarrow & G_{i+1} & \longrightarrow & G_i & \longrightarrow & G_{i-1} & \longrightarrow \ldots
\end{array}$$

This naturally gives morphisms between homologies $\phi_i : H_i(\mathcal{F}) \to H_i(\mathcal{G})$.

Definition 1.20 (Projective resolution)**.** A *projective resolution* of an R-module M is a complex of projective modules

$$\mathcal{F} : \cdots \to F_n \to \cdots \to F_1 \xrightarrow{\phi_1} F_0$$

which is exact and $\mathrm{coker}(\phi_1) = M$. Sometimes we also denote it by

$$\mathcal{F} : \cdots \to F_n \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 (\to M \to 0).$$

Definition 1.21 (Left derived functor)**.** Let T be a right-exact functor. Given a projective resolution of an R-module M:

$$\mathcal{F} : \cdots \to F_n \to \cdots \to F_1 \xrightarrow{\phi_1} F_0 (\to M \to 0).$$

Define the *left derived functor* by $L_i T(M) := H_i(T\mathcal{F})$, which is just the homology of

$$T\mathcal{F} : \cdots \to T(F_n) \to \cdots \to T(F_1) \to T(F_0) (\to T(M) \to 0).$$

We collect basic properties of derived functors here.

Proposition 1.22. (1) *$L_0 T(M) = T(M)$;*
(2) *$L_i T(M)$ is independent of the choice of projective resolution;*

(3) *If M is projective, then $L_iT(M) = 0$ for $i > 0$.*

(4) *For a short exact sequence of R-modules*

$$0 \to A \to B \to C \to 0,$$

we have a long exact sequence

$$\begin{aligned}
&\cdots \\
&\to L_3T(A) \to L_3T(B) \to L_3T(C) \\
&\to L_2T(A) \to L_2T(B) \to L_2T(C) \\
&\to L_1T(A) \to L_1T(B) \to L_1T(C) \\
&\to T(A) \to T(B) \to T(C) \to 0.
\end{aligned}$$

Definition 1.23 (Tor)**.** For an R-module N, $\mathrm{Tor}_i^R(-, N)$ is defined by $L_iT(-)$ where $T = (- \otimes N)$.

Remark 1.24. So to compute $\mathrm{Tor}_i^R(M, N)$, we should pick a projective resolution $\mathcal{F}$ of M and compute $H_i(\mathcal{F} \otimes N)$. Note that tensor products are symmetric, that is, $M \otimes N \simeq N \otimes M$, it can be seen that $\mathrm{Tor}_i^R(M, N) \simeq \mathrm{Tor}_i^R(N, M)$, and $\mathrm{Tor}_i^R(M, N)$ can be also computed by pick a projective resolution $\mathcal{G}$ of N and compute $H_i(M \otimes \mathcal{G})$.

Theorem 1.25. *TFAE:*

(1) *N is flat;*

(2) *$\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$ and all M;*

(3) *$\mathrm{Tor}_1^R(M, N) = 0$ for all M.*

Proof. (1) $\implies$ (2): take a projective resolution $\mathcal{F}$ of M, we need to compute $H_i(\mathcal{F} \otimes N)$. As N is flat, $\mathcal{F} \otimes N$ is exact, hence $\mathrm{Tor}_i^R(M, N) = 0$ for all $i > 0$.

(2) $\implies$ (3): trivial.

(3) $\implies$ (1): this follows from the long exact sequence

$$\mathrm{Tor}_1^R(M'', N) \to M' \otimes N \to M \otimes N \to M'' \otimes N \to 0.$$

□

References

[1] Atiyah, MacDonald, Introduction to commutative algebra.
[2] Eisenbud, Commutative algebra with a view toward algebraic geometry.

Shanghai Center for Mathematical Sciences, Fudan University, Jiangwan Campus, 2005 Songhu Road, Shanghai, 200438, China
E-mail address: chenjiang@fudan.edu.cn