Intro to Representation Theory

Group actions on vector spaces

- Finite groups
- Lie groups (= groups that are manifolds)
- algebraic groups = gps that are alg. variety

$\hookrightarrow$ Topic of this course

I. Background from alg geom

$\mathbb{k}$ = an algebr. closed field (much of the week: $\mathbb{k} = \mathbb{C}$)

$\mathbb{k}[T_1, \ldots, T_n]$ = poly ring in n variables

~~For any ideal $I \subset \mathbb{k}[T_1, \ldots, T_n]$~~

$\mathbb{A}^n$ = n-dim'l affine space / $\mathbb{k}$ $= \mathbb{k}^n = \{(x_1, \ldots, x_n) \mid x_i \in \mathbb{k}\}$.

For any ideal $I \subset \mathbb{k}[T_1, \ldots, T_n]$ let

$$\mathcal{V}(I) = \{\underline{x} \in \mathbb{A}^n \mid f(\underline{x}) = 0 \ \ \forall f \in I\},$$

a set obtained in this way is called an *affine variety*

Obs. Since $\mathbb{k}[T_1, \ldots, T_n]$ is noetherian, any ideal I is gen by fin many polynomials, say $f_1, \ldots, f_\ell$.

Then $\mathcal{V}(I) = \{\underline{x} \in \mathbb{A}^n \mid f_1(\underline{x}) = \cdots = f_\ell(\underline{x}) = 0\}$

The *Zariski topology* on $\mathbb{A}^n$ is the topology whose closed sets are affine varieties.

$\rightsquigarrow$ get an induced topology on any affine variety, also called Zariski top.

If $X \subset \mathbb{A}^n$, $Y \subset \mathbb{A}^m$ are affine var, then $X \times Y \subset \mathbb{A}^{n+m}$ is an affine var.

Equip with Zariski top, not product top.

A *morphism of affine varieties* $\varphi: X \to Y$ is a function that can be defined by polynomials, i.e.

$\exists\, \varphi_1, \ldots, \varphi_m \in \mathbb{k}[T_1, \ldots, T_n]$ s.t.

$\forall \underline{x} \in X$, $\varphi(\underline{x}) = \underbrace{(\varphi_1(\underline{x}), \ldots, \varphi_m(\underline{x}))}_{\in \mathbb{A}^m,\ \in Y}$.

II. Algebraic groups

Defn. An (affine) algebraic group is an affine variety G that is also a group, such that the maps

multiplication: $G \times G \longrightarrow G$
inverse : $G \longrightarrow G$

are morphisms of varieties.

Examples

1) $\mathbb{G}_a$ = "the additive group"
$= \mathbb{A}^1$, operation: $+$

addition: $\mathbb{A}^2 \longrightarrow \mathbb{A}^1$, $(x,y) \longmapsto x+y$
inverse: $\mathbb{A}^1 \longrightarrow \mathbb{A}^1$, $x \longmapsto -x$

polynomials.

2) SL_2 = 2×2 matrices with determinant 1

$$= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in \mathbb{A}^4 \;\middle|\; ad-bc-1=0 \right\}$$

multiplication: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$

inverse: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ ← polynomials

3) SL_n = $n\times n$ matrices with determinant 1 $\subset \mathbb{A}^{n^2}$

Similar to SL_2.

4) GL_n = group of all invertible $n\times n$ matrices

$$= \{ g \in \underbrace{\mathbb{A}^{n^2}}_{\text{all } n\times n \text{ matrices}} \mid \det g \neq 0 \}$$

Problems ⓐ Looks like an open set in the Zariski topology, not an affine variety

ⓑ Formula for inverse involves $\frac{1}{\det g}$ - NOT a polynomial

Repair both problems by adding a variable.

New defn:

$$GL_n = \{ (g, \delta) \in \mathbb{A}^{n^2+1} \mid \underbrace{\delta \det g - 1 = 0}_{\text{polynomial}} \}$$

g: $n\times n$ matrix, δ: $\mathbb{A}^1$

Note: $\delta = \frac{1}{\det g}$

Inversion formula: polynomial in terms of entries of g and δ

Example 5

$\mathbb{G}_m$ = the multiplicative gp
$= GL_1 = \{x \in \Bbbk \mid x \neq 0\}$
$= \{(a,\delta) \in \mathbb{A}^2 \mid a\delta - 1 = 0\}$.

Multiplication:
$(a,\delta)(a',\delta') = (aa', \delta\delta')$

Inverse: $(a,\delta)^{-1} = (\delta, a)$

III. Representations

Defn. A homomorphism of alg gps $\varphi: G \to H$ is a group homomorphism that is also a morphism of varieties.

Non-example $\Bbbk = \mathbb{C}$, $\mathbb{G}_m = \mathbb{C}^\times$

Complex conjugation $z \mapsto \bar{z}$ is a gp homom $\mathbb{C}^\times \to \mathbb{C}^\times$ but NOT a morphism of varieties.

Example. $\forall n \in \mathbb{Z}$, the map $z \mapsto z^n$ defines a homom of alg gps $\mathbb{G}_m \to \mathbb{G}_m$.
(Why is this OK for $n < 0$?)

Defn. Let G be an alg gp $/ \Bbbk$

Let V be a fin dim'l $\Bbbk$-vec sp.
(can identify $V = \mathbb{A}^n$ if you choose a basis)

$GL(V)$ = alg gp of invertible linear transformations $V \to V$

If $V = \mathbb{A}^n$ then $GL(V) = GL_n$

A (rational or algebraic) representation of G on V consists of

(Version 1) a homom of alg gps
$\pi: G \to GL(V)$

(version 2) an action of G on V,
$\rho: G \times V \to V$ such that
1) ρ is a morphism of varieties
2) $\forall g \in G$, $\rho(g, -): V \to V$ is linear.

Exercise Versions 1 & 2 are equivalent.

Representations of G are also called G-modules

Defn. Suppose we have a rep of G on V.
A subrepresentation or submodule is a linear subspace $V' \subset V$ s.t. $\forall g \in G$, $\forall v \in V'$, $g \cdot v \in V'$.

Other operations:
- quotient of a rep by subrep.
- $\oplus$ of representations
- Morphism of representations:
V, W G-modules.
A morphism of reps is a linear map $\varphi : V \to W$ s.t. $\forall g \in G$, $v \in V$, $\varphi(g \cdot v) = g \cdot \varphi(v)$.
- isomorphism of reps.

Defn. A representation V is said to be irreducible if:
1) it's nonzero, i.e. $\dim V > 0$
2) the only subreps of V are 0 & V.

Examples.
1) Any 1-dim'l rep. is automatically irreducible
2) The trivial rep of G is the rep on $\mathbb{k}$ given by
$$G \to GL_1$$
$$g \mapsto 1$$
3) The natural or defining rep of GL_n on $\mathbb{k}^n$
$$id : GL_n \to GL_n$$
4) Given an alg subgroup $H \subset G$ ($=$ closed in Zariski top on G)
& given a G-rep V,
can regard it as an H-rep.
("restriction")
Restrict the natural rep of GL_n $\rightsquigarrow$ natural rep of SL_n.

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